ON METRIC PROPERTIES OF COMPLEX POLYNOMIALS

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Let

$$f(z) = \prod_{\nu=1}^{n} (z - z_\nu) = z^n + \cdots .$$

This paper deals with metric properties of the lemniscate domain

$$E = \{ |f(z)| \leq 1\} .$$

It will give (at least partial) answers to some problems raised by Erdös, Herzog and Piranian [2]. Also, some metric properties of continua of capacity 1 will be derived.

Section 1 treats the diameters of the components of E. After some counter-examples, a lower bound for the largest diameter will be given, for the case where $|z_\nu| \leq r \leq 1$.

In Section 2 it will be proved that E contains a disk of radius $\text{const} \cdot n^{-4}$, if $z_\nu \in [-2, +2]$.

In Section 3 it will, for instance, be shown that $d \leq 4 \cdot 2^{-1/n}$ and $\Lambda < 74n^2$, where d is the measure of the projection of E onto the real axis and Λ is the perimeter of E.

Section 4 deals first with some necessary or sufficient conditions for the connectedness of E, and then with some consequences of connectedness.

The last section is concerned with the convexity of E and two related problems.

1. THE DIAMETERS OF THE COMPONENTS OF E

There is a close connection between lemniscate domains E and compact sets F with cap F = 1. Here cap F denotes the (logarithmic) capacity of F, also called the transfinite diameter of F. Every lemniscate domain $E = \{ |f(z)| \leq 1\}$ generated by $f(z) = z^n + \cdots$ has capacity 1 [4], and conversely the following approximation theorem holds [5]:

Let F be a closed bounded set with cap F = 1. Given any $\varepsilon > 0$ and $\eta > 0$, there exists a ρ $(1 < \rho < 1 + \eta)$ and a polynomial $f(z) = z^n + \cdots$ such that the lemniscate $\{ |f(z)| = \rho^n\}$ contains F in its interior and is contained in an ε-neighborhood of F.

We shall now apply the approximation theorem to some problems of Erdös, Herzog and Piranian [2]. Let E have the components E_j, of diameters d_j. Then Problem 8 asks whether

$$\sum \max(0, d_j - 1)$$

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is bounded in the class of all polynomials with highest coefficient 1. Problem 9 (revised form [2, p. 148]) asks whether the number of components with diameter greater than l is bounded in the same class, for $l > 1$ and fixed. The following theorem shows that the answer to these two questions is negative, even with $d_j - l$ $(l < 4)$ instead of $d_j - 1$ in Problem 8, and with any $l < 4$ in Problem 9.

THEOREM 1. *For each* $0 < l < 4$ *and* $k = 1, 2, \cdots$, *one can find a polynomial* $f(z) = z^n + \cdots$ *such that* $E = \{ |f(z)| \leq 1\}$ *has at least* k *different components of diameter greater than or equal to* l.

Proof. Let F be the union of the k segments $[i\mu\delta, l + i\mu\delta]$ $(\mu = 1, \cdots, k$, and $\delta > 0)$. Since the capacity of a segment is $l/4 < 1$, we have $\text{cap } F < 1$ if $\delta > 0$ is small enough (for reasons of continuity). The approximation theorem ensures the existence of a polynomial $f(z)$ such that E contains F and is contained in a $\delta/3$-neighborhood of F. Those k of the components of E that contain the k segments of F are therefore different and have diameters at least l.

I want to deal again with Problem 10 b of [2]. Let z^* be a point of E that lies on a line of support of E. Erdös, Herzog and Piranian asked whether E contains a point z with $|z - z^*| \geq 2$. I have proved [9] that this is not always true. The next theorem gives the exact constant by which 2 has to be replaced.

THEOREM 2. *If* $z^* \in E$ *lies on a line of support of* E, *there exists a point* $z \in E$ *with* $|z - z^*| > 3\sqrt{3}/4 \approx 1.299$. *The constant* $3\sqrt{3}/4$ *is best possible.*

Proof. The function

$$w = \frac{e^{-\pi i/6}(1 + z^{-1})^{2/3} + e^{\pi i/6}(1 - z^{-1})^{2/3}}{(1 + z^{-1})^{2/3} - (1 - z^{-1})^{2/3}} = \frac{3}{4}\sqrt{3}\, z + \cdots$$

maps the exterior region of the half-disk $\{ |z| \leq 1,\ \Im z \geq 0\}$ conformally onto $|w| > 1$. Hence the half-disk

$$H = \{ |z| \leq 3\sqrt{3}/4,\ \Im z \geq 0\}$$

has capacity 1. We may assume that $z^* = 0$ and that E is contained in $\Im z \geq 0$. If the theorem were not true, it would follow that $|z| \leq 3\sqrt{3}/4$ for all $z \in E$ and thus $E \subset H$. Since $\text{cap } H = 1 = \text{cap } E$, we have $E = H$. This equation can not hold, since the half-disk H is not a lemniscate domain. The approximation theorem shows that $3\sqrt{3}/4$ is the best possible constant (see [9]).

Let the zeros z_ν of $f(z)$ belong to the disk $|z| \leq r$. Erdös, Herzog and Piranian [2, Problem 7] raised the question whether there is always a component of E with diameter at least $2 - r$ $(r < 2)$. The answer is negative for $r > 1$. To show this, let $f(z) = z^n - r^n$ $(r > 1)$. Then the set $E = \{ |z^n - r^n| \leq 1\}$ has n components. If z belongs, for instance, to the component that contains the zero r and if $z^n - r^n = \omega$ $(|\omega| \leq 1)$, then for $n \to \infty$

$$|z - r| = |(r^n + \omega)^{1/n} - r| = r|1 + \omega n^{-1} r^{-n} + O(n^{-2} r^{-n}) + \cdots - 1|$$

$$\leq n^{-1} r^{-n+1}(1 + O(n^{-1})) \to 0.$$

Hence the (common) diameter of the components of E tends to 0 as $n \to \infty$. We shall need the following two lemmas to treat the case $r \leq 1$.

LEMMA 1. *Let*

$$f(z) = \prod_{\nu=1}^{n} (z - z_\nu), \qquad z_0 = \frac{1}{n}\sum_{\nu=1}^{n} z_\nu, \qquad \sigma^2 = \frac{1}{n}\sum_{\nu=1}^{n} |z_\nu|^2.$$

If $\sigma^2 - |z_0|^2 \leq 1$, *the disk*

$$|z - z_0| \leq (1 - \sigma^2 + |z_0|^2)^{1/2}$$

is contained in $E = \{|f(z)| \leq 1\}$.

Proof. Since the geometric mean is less than or equal to the arithmetic mean,

$$|f(z)|^{2/n} = \left(\prod_{\nu=1}^{n} |z - z_\nu|^2\right)^{1/n}$$

$$\leq \frac{1}{n}\sum_{\nu=1}^{n} |z - z_\nu|^2 = \frac{1}{n}\sum_{\nu=1}^{n} (|z|^2 - 2\Re[\bar{z}z_\nu] + |z_\nu|^2)$$

$$= |z|^2 - 2\Re[\bar{z}z_0] + \sigma^2$$

$$= |z - z_0|^2 - |z_0|^2 + \sigma^2,$$

and this quantity is at most 1, for $|z - z_0|^2 \leq 1 - \sigma^2 + |z_0|^2 \ (\geq 0)$.

LEMMA 2. *If* A *is a continuum,* $A \subset E$, *and* $z_\nu \in A$ *for* $\nu = 1, \cdots, n$, *then* E *is connected.*

Proof. It follows from the maximum principle that each component of E contains at least one zero. Because $z_\nu \in A \subset E$, the zeros z_ν can be connected within E, and E is itself connected.

THEOREM 3. *Let* $f(z) = \Pi(z - z_\nu)$, $|z_\nu| \leq r \leq 1$, *and let* d_0 *be the diameter of the component* E_0 *of* E *that contains* 0. *Then*

$$d_0 \geq 2 \quad \text{for} \quad 0 \leq r \leq 1/2,$$

$$d_0 > 1/r \quad \text{for} \quad 1/2 < r \leq (\sqrt{5} - 1)/2,$$

$$d_0 > 2 - r^2 \quad \text{for} \quad (\sqrt{5} - 1)/2 \leq r \leq 1.$$

Remarks. 1. Since $|f(0)| = \Pi|z_\nu| \leq 1$, the point 0 belongs to E. Therefore it is meaningful to speak of the component E_0 of E containing 0. Lemma 1 shows that the centroid z_0 lies in E_0 (compare Theorem 1 of [2]).

2. The inequality $d_0 \geq 2$ for $r \leq 1/2$ cannot be improved, as the example $f(z) \equiv z^n$ shows. Also, the polynomial

$$(z^n + 1)(z - 1)^2 (z - e^{i\pi/n})^{-1} (z - e^{-i\pi/n})^{-1}$$

has $d_0 < 1 + \varepsilon$ for sufficiently large n (see the proof of Theorem 7 in [2]). Hence the inequality $d_0 > 1$ is best possible, for $r = 1$.

3. Since all three bounds 2, r^{-1}, and $2 - r^2$ are greater than or equal to $2 - r$, Theorem 3 answers Problem 7 of Erdös, Herzog and Piranian affirmatively, for $0 < r \leq 1$.

Proof. Let $|z_\nu| \leq 1$ and $f(z) \not\equiv z^n$. We shall first prove that E_0 contains a point in $|z| > 1$. This assertion is trival if a zero z_ν with $|z_\nu| = 1$ belongs to E_0. Since E_0 contains at least one zero (by the maximum principle), we may therefore assume that at least one of the z_ν lies in $|z| < 1$, so that $|f(0)| < 1$. The polynomial

$$g(z) = \prod_{\nu=1}^{n} (1 - \bar{z}_\nu z)$$

is not constant and satisfies $|g(z)| > |f(z)|$ for $|z| < 1$ and $|g(z)| = |f(z)|$ for $|z| = 1$. Suppose that E_0 is contained in $|z| \leq 1$. Then $|g(z)| \geq |f(z)| = 1$ holds for the boundary points of E_0. Since $g(z) \neq 0$ in $|z| < 1$ and $|g(z)| = |f(z)| \neq 0$ on $\{|z| = 1\} \cap E_0$, the function $g(z)$ has no zeros in E_0. Therefore the minimum principle implies that $|g(z)| > 1$ for all interior points of E_0. But $g(0) = 1$, and 0 is an interior point of E_0, since $|f(0)| < 1$.

2. Let $0 \leq r \leq 1/2$. Then it is obvious that $|f(z)| \leq 1$ for $|z| \leq 1/2$. Hence Lemma 2 shows that E is connected. Thus $E_0 = E$, cap $E_0 = 1$, and the inequality $d_0 \geq 2$ follows from the fact that every continuum of capacity 1 has a diameter at least 2.

3. Let $1/2 < r \leq (\sqrt{5} - 1)/2$ and $z_0 = n^{-1} \Sigma z_\nu$. Since $\sigma^2 = n^{-1} \Sigma |z_\nu|^2 \leq r^2$, Lemma 1 implies that the disk

$$|z - z_0| \leq (1 - r^2 + |z_0|^2)^{1/2} \tag{1}$$

lies in E_0. If $|z_0| \leq (1 - 2r^2)/(2r)$ (> 0), then E_0 contains the disk of center 0 and radius $(1 - r^2 + |z_0|^2)^{1/2} - |z_0|$. Since this radius is a monotone decreasing function of $|z_0|$, it is not less than

$$\left(1 - r^2 + \frac{1 - 4r^2 + 4r^4}{4r^2}\right)^{1/2} - \frac{1 - 2r^2}{2r} = \frac{1}{2r} - \frac{1 - 2r^2}{2r} = r.$$

Hence the disk $|z| \leq r$ is contained in E_0, and it follows again (by Lemma 2) that $d_0 \geq 2 > 1/r$. If on the other hand $|z_0| \geq (1 - 2r^2)/(2r)$, then the radius of the disk (1) is monotone increasing, hence at least $1/(2r)$. Thus E_0 contains a disk of diameter $1/r$. Since clearly E_0 cannot be identical with this disk, it follows that $d_0 > 1/r$.

4. Finally, let $(\sqrt{5} - 1)/2 \leq r \leq 1$. It was proved in the first part that E_0 contains a point in $|z| > 1$ (except if $f(z) \equiv z^n$, in which case $d_0 = 2$). Also, the disk $|z| \leq (1 - r^2 + |z_0|^2)^{1/2} - |z_0|$ lies in E_0. Hence, for $|z_0| \leq r^2/2$,

$$\begin{aligned} d_0 &> 1 + (1 - r^2 + |z_0|^2)^{1/2} - |z_0| \\ &\geq 1 + (1 - r^2 + r^4/4)^{1/2} - r^2/2 = 2 - r^2. \end{aligned}$$

On the other hand, E_0 contains the disk (1), whose radius is at least $1 - r^2/2$ for $|z_0| \geq r^2/2$. Hence E_0 contains the disk $|z - z_0| \leq 1 - r^2/2$, but is not contained in it. Therefore $d_0 > 2 - r^2$.

2. THE LARGEST DISK CONTAINED IN E

Let A be a given compact set, let $z_\nu \in A$ ($\nu = 1, \cdots, n$), and let ρ denote the radius of the largest disk contained in E. If cap $A < 1$, there exists a positive number $\rho_0 = \rho_0(A)$ such that $\rho \geq \rho_0$ [2, Theorem 6]. If A is a disk of radius 1 or a segment of length 4 (in both cases, cap A = 1), there does not exist any positive lower bound for ρ that is independent of the degree n of f(z). Erdös, Herzog and Piranian put the question whether $\rho \geq \text{const} \cdot n^{-1}$ if $|z_\nu| \leq 1$ [2, Problem 3]. I shall only prove a weaker estimate (see also [2, Problem 2]).

THEOREM 4. *If* $|z_\nu| \leq 1$, *the lemniscate domain* E *contains a disk of radius* $(2e)^{-1} n^{-2}$.

Proof. Let again E_0 denote the component of E that contains the point 0. Theorem 3 (for r = 1) shows that the diameter of E_0 is $d_0 > 1$. Since E_0 is connected, we have $d_0 \leq 4 \text{ cap } E_0$ (see for instance [6, p. 42]), and therefore cap $E_0 > 1/4$. Since $|f(z)| \leq 1$ for $z \in E_0$, this inequality implies (see [11]) that

$$|f'(z)| \leq \frac{en^2}{2 \text{ cap } E_0} < 2en^2 \qquad (z \in E_0).$$

Let z_μ be a zero of f(z) that lies in E_0, and let z^* be the boundary point of E_0 nearest to z_μ. Taking the segment between z_μ and z^* as path of integration, we obtain

$$1 = |f(z^*)| = \left|\int_{z_\mu}^{z^*} f'(z)dz\right| < |z^* - z_\mu| \cdot 2en^2$$

and $|z^* - z_\mu| > 1/2en^2$, and therefore the disk $|z - z_\mu| \leq 1/2en^2$ is contained in $E_0 \subset E$.

Let now the given set A be the segment [-2, +2], that is, let the zeros $z_\nu = \xi_\nu$ be real, with $-2 \leq \xi_\nu \leq 2$. I want to establish the conjecture of Erdös, Herzog and Piranian [2, p. 132] that $\rho \geq n^{-\gamma}$, where γ denotes an absolute constant. We shall need

LEMMA 3. *If* $\xi = z + z^{-1}$, $\xi' = z' + z'^{-1}$, $z = x + iy$, $z' = x' + iy'$, $|z| = |z'| = 1$, $y \geq 0$, $y' \geq 0$, *then* $|\xi - \xi'| \geq |z - z'|^2$.

Proof. $|\xi - \xi'| = |z - z'|\,|1 - (zz')^{-1}| = |z - z'|\,|z - \bar{z}'| \geq |z - z'|^2$, because

$$|z - \bar{z}'|^2 = (x - x')^2 + (y + y')^2 \geq (x - x')^2 + (y - y')^2 = |z - z'|^2.$$

THEOREM 5. *Let* $-2 \leq z_\nu = \xi_\nu \leq 2$. *Then the set* $E \cap X$ *contains a segment of length* $1/8e^2n^4$ (X denotes the real axis).

Proof. Let

$$g(z) = z^n f(z + z^{-1}) = \prod_{\nu=1}^{n} (z^2 - \xi_\nu z + 1),$$

which is a polynomial of degree 2n. The zeros $\xi_\nu/2 \pm i(1 - \xi_\nu^2/4)^{1/2}$ of g(z) have absolute value 1. The proof of Theorem 4 shows that the set $\{|g(z)| \leq 1\}$ contains a disk of radius $1/2en^2$ and center on $|z| = 1$. We can thus choose an arc B on

$|z| = 1$, of diameter $1/4en^2$, such that $|g(z)| \leq 1$ on B, and such that $\Im z$ has always the same sign on B, say $\Im z \geq 0$. The segment $B^* = \{\xi = z + z^{-1}: z \in B\}$ has length at least $(4en^2)^{-2}$, by Lemma 3, and we have

$$|f(\xi)| = |z^n f(z + z^{-1})| \leq 1$$

for $\xi \in B^*$, hence $B^* \subset E \cap X$.

3. UPPER BOUNDS FOR GEOMETRIC QUANTITIES ASSOCIATED WITH E

Pólya [8] has proved that the linear measure d of the projection of a compact set with cap F = 1 onto a straight line satisfies $d \leq 4$. I want to give the exact upper bound of d for lemniscate domains $E = \{|f(z)| \leq 1\}$ of polynomials of degree n.

THEOREM 6. *Let* $f(z) = \Pi_{\nu=1}^{n} (z - z_\nu)$, *let* P *be the projection of* E *onto the real axis* X, *and let* d *be the linear measure of* P. *Then*

$$\text{cap } P \leq 2^{-1/n}, \qquad d \leq 4 \cdot 2^{-1/n},$$

with cap $P = 2^{-1/n}$ *exactly if all* z_ν *lie on a parallel to* X *and if* $\mathring{E} = \{|f(z)| < 1\}$ *has* n *components.* (The result concerning d was already known to P. Erdös and Bl. Sendov; see the remark after Problem 102, Wisk. Opgaven 20/3 (1957), p. 22.)

Remark. The equation $d = 4 \cdot 2^{-1/n}$ holds exactly if cap $P = 2^{-1/n}$ and P is one segment. Then $\mathring{E}$ consists of n components whose boundaries meet in pairs at the n - 1 zeros of $f'(z)$. It can be shown that these conditions are satisfied if and only if $f(z) = T_n(2^{-1+1/n}z + c)$, where $T_n(\zeta)$ is the n-th Tchebycheff polynomial and c is a complex constant.

Proof. 1. Let $z_\nu = x_\nu + iy_\nu$ and

$$f^*(z) = \prod_{\nu=1}^{n} (z - x_\nu), \qquad E^* = \{|f^*(z)| \leq 1\}.$$

If x is a point of the projection P, then $z = x + iy \in E$ for a certain y, hence

$$|f^*(x)| = \Pi|x - x_\nu| \leq \Pi|z - z_\nu| = |f(z)| \leq 1.$$

Therefore we have

$$P \subset E^* \cap X. \tag{2}$$

Suppose that $P = E^* \cap X$. If x' is the greatest value in $E^* \cap X = P$, then, for a certain y' and for $z' = x' + iy'$, $1 = |f^*(x')| \leq |f(z')| \leq 1$, hence $|f^*(x')| = |f(z')|$ and

$$\Pi((x' - x_\nu)^2 + (y' - y_\nu)^2) = \Pi(x' - x_\nu)^2,$$

and therefore $y' - y_\nu = 0$ for $\nu = 1, \cdots, n$.

2. Let $R = \{z: f^*(z) \text{ real}, |f^*(z)| \leq 1\}$. Since the segment [-1, +1] has capacity 1/2, a theorem of Fekete [4] (see for instance [6, p. 259]) shows that cap $R = 2^{-1/n}$. Since $f^*(x)$ is real for real x, $E^* \cap X \subset R$. Therefore, by (2),

(3) $$\operatorname{cap} P \leq \operatorname{cap}(E^* \cap X) \leq \operatorname{cap} R = 2^{-1/n}$$

and [8]

$$d \leq 4 \operatorname{cap} P \leq 4 \cdot 2^{-1/n}.$$

3. We have $\operatorname{cap} P = 2^{-1/n}$ exactly if the sign of equality stands in all inequalities (3), hence if and only if $P = E^* \cap X = R$. Part 1 of this proof shows that $P = E^* \cap X$ holds exactly if the zeros z_ν lie on a parallel to X, which we may assume to be X itself. Then this means $f^*(z) \equiv f(z)$. Suppose that $\mathring{E}$ has the maximal number n of components. Each component is mapped by $w = f(z)$ onto $|w| < 1$. Because f(z) assumes every value only n times, f(z) is real in $\mathring{E}$ only for real z. Hence $E^* \cap X = E \cap X = R$. On the other hand, suppose that $\mathring{E}$ has fewer than n components. Then there exists a real $\xi \in \mathring{E}$ with $f'(\xi) = 0$. A certain small curve that begins in ξ and goes into $\Im z > 0$ is consequently mapped by $w = f(z)$ into the real axis. Hence $E \cap X$ is properly contained in R, and $\operatorname{cap} P < 2^{-1/n}$. Thus Theorem 6 is proved.

The inequality $d \leq 4$ implies that the maximum of the measures of the different projections of E is at most 4. Let b be the minimum of the measures of the projections of E. By applying the approximation theorem to the "5-Stern" [10, p. 73] we obtain a lemniscate domain with $b > 2.386$ (compare [2, Problem 10a]). I shall give an upper bound for b.

THEOREM 7. *Let* F *be a closed bounded set with* $\operatorname{cap} F = 1$. *Then the projection of* F *onto a certain straight line has measure less than* 3.30.

Proof. The set F can be enclosed by a system of closed curves L_μ ($\mu = 1, \cdots, m$) whose lengths Λ_μ satisfy

$$\sum_{\mu=1}^{m} \Lambda_\mu < 10.36$$

[12, Theorem 2]. We may assume that these curves are convex (otherwise we take instead the boundaries of their convex hulls; if these intersect, we take them as one curve, and so forth). Let $b_\mu(\theta)$ be the width of L_μ in the direction θ, that is, the width of the narrowest strip containing L_μ that forms the angle θ with the real axis. Then [1, p. 48]

$$\Lambda_\mu = \frac{1}{2}\int_0^{2\pi} b_\mu(\theta)\, d\theta,$$

and therefore

$$\frac{1}{2}\int_0^{2\pi} \sum_{\mu=1}^{m} b_\mu(\theta)\, d\theta = \sum_{\mu=1}^{m} \Lambda_\mu < 10.36.$$

This inequality implies that

$$\sum_{\mu=1}^{m} b_\mu(\theta_0) = \min_\theta \sum_{\mu=1}^{m} b_\mu(\theta) < 10.36/\pi < 3.30.$$

The projection of F onto the straight line of direction $\theta_0 + \pi/2$ has measure at most $\Sigma b_\mu(\theta_0) < 3.30$.

We shall now consider the linear measure λ of the intersection of E with the unit circle $|z| = 1$. The equation $\lambda = 2\pi$ holds if and only if $f(z) = z^n$. Even if the zeros z_ν satisfy $|z_\nu| = 1$, the measure λ can come arbitrarily near to 2π. I shall give an upper estimate for λ that depends only on the degree n. This estimate shows that λ does not depend continuously on the set $\{z_1, \cdots, z_\nu\}$.

THEOREM 8. *Let* $f(z) \not\equiv z^n$. *Then*

$$\text{meas}[E \cap \{|z| = 1\}] \leq 2\pi \frac{n}{n+1}.$$

Proof. We write $f(z) = \Sigma_{k=0}^n a_k z^k$ and $\alpha = 2\pi/(n+1)$. Then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |f(e^{i\theta + i\alpha\nu})|^2 = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sum_{\nu=0}^{n} a_k \bar{a}_\ell e^{i(k-\ell)} e^{i\alpha\nu(k-\ell)}$$

$$= 1 + |a_{n-1}|^2 + \cdots + |a_0|^2.$$

Since $f(z) \not\equiv z^n$, this quantity is greater than 1. Hence there exists, for each θ in $0 \leq \theta < \alpha$, an integer μ $(0 \leq \mu \leq n)$ such that $|f(e^{i\theta + i\mu\alpha})| > 1$. Let $\mu(\theta)$ be the least of these integers, and let M denote the set of values ϕ in $[0, 2\pi)$ which have the form $\phi = \theta + \alpha\mu(\theta)$ $(0 \leq \theta < \alpha)$. Then M has measure $\alpha = 2\pi/(n+1)$, and the theorem follows from the fact that the set $M \cap E$ is empty.

Finally, I shall obtain an upper bound for the length Λ of the lemniscate $\{|f(z)| = 1\}$. Problem 12a in [2] asks whether Λ is greatest for $f(z) = z^n - 1$. An affirmative answer would imply that $\Lambda \leq 2n + o(n)$.

THEOREM 9. *If* $f(z) = z^n + \cdots$ *and* Λ *is the length of* $C = \{|f(z)| = 1\}$, *then* $\Lambda < 74n^2$.

Proof. The lemniscate C is given by the algebraic equation

$$f(z)\overline{f(z)} = 1 \qquad (z = x + iy,\ x \text{ and } y \text{ real})$$

with real coefficients. Hence C is the "real" part of a plane algebraic curve of order 2n, "real" in the sense of algebraic geometry, that is with real x and y. We may assume, for reasons of continuity, that this curve has only simple singularities and no "real" double points. Then C has at most $2n(2n-2)$ "real" points of inflection [7], because C does not have any "real" cusps and isolated points. At the points at which C has a tangent parallel to the real axis X, an algebraic equation of degree $2n - 1$ is satisfied. Since it is easily seen that there are only finitely many such points, the theorem of Bézout shows that C has at most $2n(2n-1)$ points with a tangent parallel to X.

We mark on C all points of inflection and all points with a tangent parallel to X. The number of these points together is less than $8n^2$. They divide the lemniscate C into simple arcs C_k $(k = 1, \cdots, m)$ on each of which the curvature has constant sign, and which have no interior points with a tangent parallel to X. It is easy to see that joining the endpoints of an arc C_k by a segment gives a closed convex curve C_k^*. The endpoints of every arc C_k belong to one of the two categories just described. The number m of arcs therefore satisfies the condition $m < 8n^2$. Because $C_k \subset C$, we have $\operatorname{cap} C_k \leq \operatorname{cap} C = 1$, and the convex hull of C_k has a perimeter less than 9.2 [10, Theorem 5]. Therefore the length Λ_k of C_k is

$$\Lambda_k \leq \text{length of } C_k^* < 9.2$$

and

$$\Lambda = \sum_{k=1}^{m} \Lambda_k < 9.2m < 9.2 \cdot 8n^2 < 74n^2 .$$

4. THE CONNECTEDNESS OF E

Let $f(z) = \Pi_{\nu=1}^{n} (z - z_\nu)$. We shall first obtain some relations between the connectedness of the set $E = \{|f(z)| \leq 1\}$ and the distribution of the zeros z_ν. We shall need

LEMMA 4. *Let* $C(r) = \{z: |f(z)| = r^n\}$ $(r \geq 0)$ *and*

$$\lambda(r) = \frac{1}{2\pi r^n} \int_{C(r)} |z|^2 \, |f'(z)| \, |dz| . \tag{4}$$

Then

$$\lambda(r) > \sum_{\nu=1}^{n} |z_\nu|^2$$

for $r > 0$.

Proof. 1. Let $r > 0$ be a value for which C(r) does not contain any zero of the derivative f'(z). Then there exist integers m and p_k such that C(r) consists of m closed analytic curves each of which is mapped p_k times $(k = 1, \cdots, m;$ $p_1 + \cdots + p_m = n)$ onto the circle $|w| = r^n$, by the function $w = f(z)$. Each function $w_k = f(z)^{1/p_k}$ is therefore regular on one of these curves and maps it one-to-one ont onto $|w| = r^{n/p_k}$. Let $z = \phi_k(w_k)$ $(k = 1, \cdots, m)$ denote the inverse function. Then

$$w_k^{p_k} = f(\phi_k(w_k)), \qquad p_k w_k^{p_k - 1} = f'(z)\, \phi_k'(w_k) ,$$

and therefore, by (4),

$$\begin{aligned} \lambda(r) &= \sum_{k=1}^{m} \frac{p_k}{2\pi r^n} \int_{|w_k| = r^{n/p_k}} |\phi_k(w_k)|^2 \, |w_k|^{p_k - 1} \, |dw_k| \\ &= \sum_{k=1} \frac{p_k}{2\pi} \int_0^{2\pi} |\phi_k(r^{n/p_k} e^{i\theta})|^2 \, d\theta . \end{aligned} \tag{5}$$

Differentiation gives

$$\lambda'(r) = \sum_{k=1}^{m} \frac{p_k}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}[\phi_k(r^{n/p_k}e^{i\theta})\bar{\phi}_k]\,d\theta$$

$$= \sum_{k=1}^{m} \frac{p_k}{2\pi}\int_0^{2\pi} \frac{n}{p_k} r^{n/p_k - 1}\, 2\,\Re[e^{i\theta}\phi_k'(r^{n/p_k}e^{i\theta})\cdot\bar{\phi}_k]\,d\theta$$

$$= \sum_{k=1}^{m} \frac{n}{\pi r}\int_0^{2\pi} \Re\left[\frac{1}{i}\frac{\partial}{\partial\theta}\phi(r^{n/p_k}e^{i\theta})\cdot\bar{\phi}_k\right]d\theta .$$

We write $\phi_k(r^{n/p_k}e^{i\theta}) = u_k(\theta) + iv_k(\theta)$ and obtain

$$\lambda'(r) = \sum_{k=1}^{m} \frac{n}{\pi r}\int_0^{2\pi} (v_k' u_k - u_k' v_k)\,d\theta .$$

Since $u_k + iv_i$ represents a positively orientated simple closed curve, the value of the integral is twice the area enclosed by this curve, and is therefore positive. Hence $\lambda'(r) > 0$ for all r, except possibly for a finite number of values. Since $\lambda(r)$ is continuous, this function is strictly increasing.

2. If $r > 0$ is sufficiently small, then $f'(z) \neq 0$ within $C(r)$, except at the multiple zeros of $f(z)$. We now denote the multiplicities of the zeros by p_k. Temporarily, we may relabel the zeros in such a way that $f(z)$ has m different p_k-fold zeros z_k $(k = 1, \cdots, m)$. Let again $z = \phi_k(w_k)$ be the inverse function of

$$w_k = f(z)^{1/p_k} = c_k(z - z_k) + \cdots \qquad (c_k \neq 0).$$

Then $\phi_k(0) = z_k$, equation (5) is again applicable, and it follows that

$$\lim_{\rho\to 0} \lambda(\rho) = \sum_{k=1}^{m} p_k|\phi_k(0)|^2 = \sum_{k=1}^{m} p_k|z_k|^2 = \sum_{\nu=1}^{n} |z_\nu|^2 .$$

Since $\lambda(\rho)$ is strictly increasing,

$$\lambda(r) > \lim_{\rho\to 0}\lambda(\rho) = \sum_{\nu=1}^{n} |z_\nu|^2 .$$

THEOREM 10. *Let*

$$z_0 = \frac{1}{n}\sum_{\nu=1}^{n} z_\nu = 0 \qquad \textit{and} \qquad \sigma^2 = \frac{1}{n}\sum_{\nu=1}^{n}|z_\nu|^2 .$$

Then the following best possible results hold:

(a) *If* $|z_\nu| \leq \sqrt{2}/2$ *or if* $z_\nu \in [-1, +1]$, *then* E *is connected.*

(b) *If* E *is connected, then* $|z_\nu| < 2$ *and* $\sigma < \sqrt{2}$.

Proof. (a) Let $|z_\nu| \leq \sqrt{2}/2$. Because $z_0 = 0$, Lemma 1 implies that the disk $|z| \leq \sqrt{2}/2$ is contained in E. Hence E is connected, by Lemma 2. The polynomial $(z^2 - 1/2)^m (z^2 + a^2)$ with $a > \sqrt{2}/2$ has three distinct components, if m is sufficiently large. Hence the bound $\sqrt{2}/2$ cannot be improved.

Let z_ν be contained in the segment $[-1, +1]$. Then both halves $[-1, 0]$ and $[0, 1]$ lie in E [2, Theorem 1], and Lemma 2 shows again that E is connected. The polynomial $z^2 - a^2$ with $a > 1$ has two components.

(b) Let E be connected. Then (because $z_0 = 0$)

$$w = f(z)^{1/n} = (z^n + a_{n-2} z^{n-2} + \cdots)^{1/n} = z + a_2^* z^{-1} + \cdots. \tag{6}$$

Since this function is univalent in the exterior region $\{|f(z)| > 1\}$ of E, the inverse function

$$z = \phi(w) = w + \sum_{\mu=1}^{\infty} b_\mu w^{-\mu} \tag{7}$$

is meromorphic and univalent in $|w| > 1$. Hence E is contained in $|z| \leq 2$ (see for instance [6, p. 42]), and it follows that $|z_\nu| < 2$ because z_ν is an interior point of E. Using (5) (with $m = 1$, $p_1 = n$) and (7), we obtain for $r > 1$

$$\frac{\lambda(r)}{n} = \frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^2 \, d\theta = r^2 + \sum_{\mu=1}^{\infty} |b_\mu|^2 r^{-2\mu},$$

and therefore, by Lemma 4,

$$\sigma^2 = \frac{1}{n}\sum_{\nu=1}^{n} |z_\nu|^2 < \frac{\lambda(1)}{n} = 1 + \sum_{\mu=1}^{\infty} |b_\mu|^2.$$

Since $\phi(w)$ is univalent in $|w| > 1$, the area theorem [6, p. 39]

$$\sum_{\mu=1}^{\infty} \mu |b_\mu|^2 \leq 1$$

gives $\sigma^2 < 2$.

The lemniscate domain E of the polynomial

$$T_n(2^{1/n-1} z) = \cos[n \arccos(2^{1/n-1} z)] = z^n + \cdots$$

is connected, and the zeros are

$$x_\nu^{(n)} = 2^{1-1/n} \cos\frac{\pi}{2n}(2\nu - 1) \qquad (\nu = 1, \cdots, n).$$

The zero $x_1^{(n)} = 2^{1-1/n}\cos\frac{\pi}{2n}$ tends to 2 as $n \to \infty$, and

$$\frac{1}{n}\sum_{\nu=1}^{n} x_\nu^{(n)2} = 2^{2-2/n}\cdot\frac{1}{n}\sum_{\nu=1}^{n}\cos^2\frac{\pi}{2n}(2\nu-1)$$

$$\to \frac{4}{\pi}\int_0^{\pi}\cos^2 t\,dt = \frac{4}{\pi}\frac{\pi}{2} = 2\,.$$

Hence the bounds in (b) cannot be improved.

The last sufficient condition in Theorem 10a is $z_0 = 0$ and $z_\nu \in [-1, +1]$. The segment $[-1, +1]$ has capacity $1/2$. To generalize the condition on the z_ν, we need the following lemma.

LEMMA 5. *If* K *is the convex hull of the zeros* z_ν, *then*

$$g(z) = f(z)^{1/n} = z + \cdots$$

is univalent in the exterior region of K.

Proof. Let L be an arbitrary convex analytic curve that contains K in its interior and is positively orientated. We assert that

$$\arg g(z) = n^{-1}\arg f(z)$$

increases monotonically on L. It is enough to prove that $\arg f(z)$ increases on each orientated straight line that leaves all z_ν on its left side. We may assume this line to be the imaginary axis. Then

$$\arg f(iy) = \sum_{\nu=1}^{n} \Im\,[\log(iy - x_\nu - iy_\nu)]\,,$$

where $z_\nu = x_\nu + iy_\nu$ and $x_\nu < 0$, and therefore

$$\frac{d}{dy}\arg f(iy) = \sum_{\nu=1}^{n}\Im\left[\frac{i}{i(y-y_\nu) - x_\nu}\right]$$

$$= \sum_{\nu=1}^{n}\frac{-x_\nu}{(y-y_\nu)^2 + x_\nu^2} > 0\,,$$

which was to be proved. The variation of $\arg f(z)$ on L is $2\pi n$, by the argument principle. Hence the variation of $\arg g(z)$ is 2π, and $\arg g(z)$ increases monotonically by 2π on L. Therefore $g(z)$ is univalent on L and consequently in the entire exterior region of K.

THEOREM 11. *Let* A *be a closed bounded convex set with* cap $A = \kappa \leq 1/2$ *and conformal center* 0. *This means that the function* $\psi(w)$ *that maps* $|w| > 1$ *conformally onto the exterior region of* A *has the development*

$$\psi(w) = \kappa w + c_1 w^{-1} + \cdots\,.$$

If the zeros z_ν *of* $f(z)$ *belong to* A *and if their centroid* z_0 *is* 0, *then* E *is connected and contains* A.

Proof. Since the convex set A contains the zeros z_ν and therefore their convex hull K, the function $g(z) = f(z)^{1/n}$ is univalent in the exterior region of A (Lemma 5). Hence, by equation (6),

$$g(\psi(w)) = \kappa w + c_1^* w^{-1} + \cdots,$$

and $g(\psi(w))$ is univalent in $|w| > 1$. Therefore

$$\max_{|w|=1} |g(\psi(w))| \leq 2\kappa \leq 1$$

[6, p. 32], and it follows that

$$\max_{z\in A} |f(z)| = \max_{z\in A} |g(z)|^n \leq 1.$$

This inequality means that $A \subset E$, and Lemma 2 shows that E is connected.

We shall now derive some metric properties of E for the case where E is connected. More generally, we shall consider a continuum F of capacity 1. Let b and d be the width and the diameter of F. Erdös, Herzog and Piranian put the problem to find bounds for b, d, bd and related quantities [2, Problem 15]. I have proved the (not best possible) inequality $b < 2.920$ [10, Theorem 6].

In another note [9], I asserted that $b^2 + d^2 \leq 63/3$; but the proof was not correctly formulated, as Prof. Herzog kindly pointed out to me. I want to complete the proof here: Inequalities (3) and (4) of [9] imply

$$b^2 \leq -\frac{32}{3} - 16\beta + \frac{64}{3}\left(1 + \frac{3}{4}\beta\right)^{1/2}$$

$$d^2 \leq -\frac{32}{3} + 16\beta + \frac{64}{3}\left(1 - \frac{3}{4}\beta\right)^{1/2},$$

where $0 \leq \beta \leq 1$. Hence

$$b^2 + d^2 \leq \frac{64}{3}\left[\left(1 + \frac{3}{4}\beta\right)^{1/2} + \left(1 - \frac{3}{4}\beta\right)^{1/2} - 1\right] \leq \frac{64}{3},$$

which is the asserted inequality.

Probably $b^2 + d^2 \leq 16$ holds (with equality for a segment of length 4). For the case where F is convex or contains at least a segment of length d, the inequality $b^2 + d^2 \leq 16$ has been proved [10, Theorem 9].

THEOREM 12. *Let* F *be a continuum, of capacity* 1 *and symmetric with respect to the point* 0. *Then either*

(I) $b \leq 2\sqrt{2}$, $b^2 + d^2 \leq 16$ *and* $bd \leq 8$, *or*

(II) $b \leq 2$, $b^2 + d^2 \leq 18$ *and* $bd \leq 4\sqrt{3}$.

Remarks. 1. There exists a symmetric continuum of capacity 1 with $b > 2.18$, and one with $bd > 6.15$ [9].

2. Let $z^* \in F$ be a point with the maximal distance $d/2$ from 0, and let b^* be the width of the narrowest strip that contains F and is parallel to the diameter $[-z^*, z^*]$ of F. Using the method of the proof which follows, one can show that

(8) $$b^* \le 2\sqrt{2}, \qquad b^* d \le 8.$$

The example $F = [-\sqrt{2}, \sqrt{2}] \cup [-i\sqrt{2}, i\sqrt{2}]$ and $z^* = \sqrt{2}$ has $b^* = 2\sqrt{2}$, $b^* d = 8$. Hence the inequalities (8) are best possible.

Proof. Let $h(w) = w + \beta w^{-1} + \cdots$ be the odd function that maps $|w| > 1$ conformally onto the exterior region of F. We may assume that β is real and nonnegative. Then $0 \le \beta \le 1$. The function

$$h(w^{1/2})^2 = w + 2\beta + \cdots$$

is meromorphic and univalent in $|w| > 1$. If z is a point of F, then $h(w) \ne \pm z$ and $h(w^{1/2})^2 \ne z^2$, in $|w| > 1$. Therefore $|z^2 - 2\beta| \le 2$ [6, p. 42]. Hence F is contained in

$$|(z/\sqrt{2})^2 - \beta| \le 1.$$

Elementary computations show that this inequality implies

$$b \le \begin{cases} 2\sqrt{2}(1-\beta)^{1/2} & \text{for } 0 \le \beta \le 1/2, \\ \sqrt{2}\beta^{-1/2} & \text{for } 1/2 \le \beta \le 1, \end{cases}$$

and

$$d \le 2\sqrt{2}(1+\beta)^{1/2}.$$

In the case (I) where $0 \le \beta \le 1/2$, we have therefore $b^2 + d^2 \le 16$, $bd \le 8$, $b \le 2\sqrt{2}$, and in the case (II) where $1/2 \le \beta \le 1$, we have $b \le 2$ and

$$b^2 + d^2 \le 2\beta^{-1} + 8(1+\beta) \le 18, \qquad bd \le 4(1+\beta^{-1})^{1/2} \le 4\sqrt{3}.$$

5. CONVEXITY

Let $f(z) = \Pi(z - z_\nu)$, and let $z_0 = \frac{1}{n}\sum z_\nu$. Erdös, Herzog and Piranian [2, Theorem 11] have proved that the set $E = \{|f(z)| \le 1\}$ is convex if

$$|z_\nu| \le \frac{\sin \pi/8}{1 + \sin \pi/8} \approx 0.277.$$

The following theorem improves this result slightly.

THEOREM 13. *If one of the conditions*

$$\text{(a) } |z_\nu| \le 0.320, \qquad \text{(b) } |z_\nu| \le 0.424 \text{ and } z_0 = 0$$

is satisfied, then E *is convex.*

Proof. Let $\rho > 1$, and write $\zeta_\nu = \rho z_\nu$ and

(9) $$g(\zeta) = \prod_{\nu=1}^{n} (\zeta - \zeta_\nu) = \prod_{\nu=1}^{n} (\zeta - \rho z_\nu) = \rho^n f(\rho^{-1}\zeta).$$

(a) Let $|\zeta_\nu| \leq 1/2$. Then $|g(\zeta)| \leq 1$ for $|\zeta| \leq 1/2$, and Lemma 2 shows that $F = \{|g(\zeta)| \leq 1\}$ is connected. Lemma 1 implies that the disk

$$|\zeta - \zeta_0| \leq \left(1 - \frac{1}{4}\right)^{1/2} = \frac{1}{2}\sqrt{3}$$

(with $\zeta_0 = \rho z_0$) is contained in F. The area of F is therefore at least $3\pi/4$. Let

$$\zeta = \psi(\omega) = \omega + \sum_{\mu=0}^{\infty} b_\mu \omega^{-\mu}$$

be the inverse function to $\omega = g(\zeta)^{1/n}$. Since $\psi(\omega)$ is univalent in $|\omega| > 1$, the area of F is

$$\pi\left(1 - \sum_{\mu=1}^{\infty} \mu |b_\mu|^2\right),$$

hence $\Sigma \mu |b_\mu|^2 \leq 1/4$. Applying the Schwarz inequality, we obtain for $\rho > 1$

$$\left(\sum_{\mu=1}^{\infty} \mu^2 |b_\mu| \rho^{-(\mu+1)}\right)^2 \leq \sum_{\mu=1}^{\infty} \mu |b_\mu|^2 \cdot \sum_{\mu=1}^{\infty} \mu^3 \rho^{-2(\mu+1)}$$

$$\leq \frac{1}{4}\rho^{-4} \frac{1 + 4\rho^{-2} + \rho^{-4}}{(1 - \rho^{-2})^4}.$$

If we put $\rho^{-2} = 0.41$, the last term is less than 1, and therefore

$$\sum_{\mu=1}^{\infty} \mu^2 |b_\mu| \rho^{-(\mu+1)} < 1.$$

This inequality implies that the curve $\{\zeta = \psi(\omega): |\omega| = \rho\}$ $(\rho^{-2} = 0.41)$ is convex (see Hilfssatz 4b in [13]). The curve can be written $\{|g(\zeta)| = \rho^n\}$, or, by (9), $\{\zeta: |f(\rho^{-1}\zeta)| = 1\}$. Therefore the set $E = \{z: |f(z)| \leq 1\}$ is convex if

$$|z_\nu| = \rho^{-1} |\zeta_\nu| \leq 0.41^{1/2} \cdot 0.5 > 0.32.$$

(b) Let $|\zeta_\nu| \leq \sqrt{2}/2$. Then $\zeta_0 = \rho z_0 = 0$, and the set $F = \{|g(\zeta)| \leq 1\}$ contains the disk $|\zeta| \leq \sqrt{2}/2$ (Lemma 1). Hence F is connected, and its area is at least $\pi/2$. Therefore

$$\left(\sum_{\mu=1}^{\infty} \mu^2 |b_\mu| \rho^{-(\mu+1)}\right)^2 \leq \frac{1}{2} \cdot \rho^{-4} \frac{1 + 4\rho^{-2} + \rho^{-4}}{(1 - \rho^{-2})^4},$$

and this quantity is less than 1 for $\rho^{-2} = 0.36$. Hence E is convex if

$$|z_\nu| = \rho^{-1} |\zeta_\nu| \leq 0.6 \cdot \sqrt{2}/2 > 0.424.$$

Let $f(z) = \Pi_{k=1}^{m} (z - z_k)^{p_k}$ (z_k distinct, p_k positive integers), and let

$E = \{|f(z)| \leq 1\}$ have the maximal number m of components. H. Grunsky (see [2, Problem 16]) raised the question whether all components must be convex. I shall give a counter-example.

THEOREM 14. *Let* $f(z) = z^p(z - a)$. *If* $a - (1 + p^{-1}) \cdot p^{1/(p+1)}$ *is positive and sufficiently small, and* p *is sufficiently large, then the set* $E = \{|f(z)| \leq 1\}$ *has two components, one of which is not convex.*

Proof. We put $\xi = p^{1/(p+1)}$ and $f_p(z) = z^p(z - (1 + p^{-1})\xi)$. Then

$$\frac{f_p'(z)}{f_p(z)} = \frac{p}{z} + \frac{1}{z - (1 + p^{-1})\xi},$$

$$f_p(\xi) = -\xi^p \cdot p^{-1}\xi = -1,$$

and therefore

$$f_p'(\xi) = -\frac{p}{\xi} + \frac{p}{\xi} = 0.$$

Thus ξ is a double-point of the curve $C_p = \{|f_p(z)| = 1\}$, and $E_p = \{|f_p(z)| \leq 1\}$ consists of two parts which have only the one common point ξ. The two branches of C_p in ξ have the tangents $y = \pm(x - \xi)$. Hence $E_p \cap \{|z - \xi| \leq \delta\}$ is contained in the set

$$S = \{x + iy: |y| \leq 1.1 \cdot |\xi - x|\},$$

for some small $\delta > 0$. The point $z^* = 0.5 + i \cdot 0.6$ ($|z^*| < 1$) satisfies $f_p(z^*) \to 0$ as $p \to \infty$. It follows that $z^* \in E_p$ for large p. Also, $z^* \notin S$ for large p, since $\xi \to 1$ as $p \to \infty$. Thus the segment connecting z^* and ξ contains a point that does not belong to E_p, and therefore the part of E_p containing 0 is not convex for large p. If $a - (1 + p^{-1})\xi$ is positive and sufficiently small, then $E = \{|f(z)| \leq 1\}$ has two components, and the component that contains the point 0 is not convex.

Finally I shall deal with the following problem of Erdös, Herzog and Piranian [2, Problem 13]. Let z_ν be n complex numbers which satisfy $|z_\mu - z_\nu| \leq 2$ ($\mu, \nu = 1, \cdots, n$). Is $\Pi_{\nu=1} \Pi_{\mu \neq \nu} |z_\mu - z_\nu|$ maximal if the z_ν are the vertices of a regular n-gon of diameter 2? We denote the maximum by Δ_n:

$$\Delta_n = \max_{\substack{z_1, \cdots, z_n \\ |z_\mu - z_\nu| \leq 2}} \prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu|. \tag{10}$$

The conjecture implies that $\Delta_n = n^n$ for even n and $\Delta_n = n^n(\cos \pi/2n)^{-n(n-1)}$ for odd n. The last quantity is

$$n^n\left(1 - \frac{\pi^2}{8n^2} + \cdots\right)^{-n(n-1)} \sim n^n e^{\pi^2/8}.$$

In order to obtain an estimate for Δ_n, we need a result on convex sets.

LEMMA 6. *Let* K *be a convex continuum of capacity* 1. *Then there exist polynomials*

$$f_n(z) = z^n + \cdots \qquad (n = 1, 2, \cdots)$$

with zeros in K *such that* $\max_{z \in K} |f_n(z)| \leq 4$.

Proof. Let $\psi(w) = w + \cdots$ be the function that maps $|w| > 1$ conformally onto the exterior region of K. We shall prove that

$$f_n(z) = \prod_{\nu=1}^{n} (z - \psi(e^{2\pi i\nu/n}))$$

satisfies $|f_n(z)| \leq 4$ for $z \in K$. Let z be a fixed point of K, and let

$$\Phi(w) = e^{-\pi i(n+1)/n} \prod_{\nu=1}^{n} (\psi(e^{2\pi i\nu/n} w) - z)^{1/n} = w + \cdots,$$

for $|w| > 1$. Then

$$\Re\left(w \frac{\Phi'(w)}{\Phi(w)}\right) = \frac{1}{n} \sum_{\nu=1}^{n} \Re\left(e^{2\pi i\nu/n} w \frac{\psi'(e^{2\pi i\nu/n} w)}{\psi(e^{2\pi i\nu/n} w) - z}\right). \tag{11}$$

A convex set is star-like with respect to each point in it. Hence $\psi(e^{2\pi i\nu/n} w) - z$ maps $|w| > 1$ onto the complement of a continuum which is star-like with respect to the point 0, so that every term of the sum in (11) is positive. Therefore $\Phi(w)$ is also a star-like univalent function that does not vanish. Furthermore,

$$\Phi(e^{2\pi i/n} w) = e^{-\pi i(n+1)/n} \prod_{\nu=1}^{n} (\psi(e^{2\pi i(\nu+1)/n} w) - z)^{1/n} = e^{2\pi i/n} \Phi(w).$$

It follows that the function

$$\Psi(w) = \Phi(w^{1/n})^n = w + \cdots$$

is again meromorphic, univalent, and different from 0 in $|w| > 1$. Hence $\max_{|w|=1} |\Psi(w)| \leq 4$, and

$$|f_n(z)| = \prod_{\nu=1}^{n} |\psi(e^{2\pi i\nu/n}) - z| = |\Psi(1)| \leq 4.$$

THEOREM 15. *Let* K *be a convex continuum of capacity* 1. *Then, for* $z_\nu \in K$ $(\nu = 1, \cdots, n)$,

$$\prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu| \leq 2^{4(n-1)} n^n.$$

Proof. Using well-known properties of determinants, we obtain

$$\prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu| = \left| \begin{vmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_2 & \cdots & z_n^{n-1} \end{vmatrix} \right|^2$$

$$= \left| \begin{vmatrix} 1 & f_1(z_1) & \cdots & f_{n-1}(z_1) \\ \vdots & \vdots & & \vdots \\ 1 & f_1(z_n) & \cdots & f_{n-1}(z_n) \end{vmatrix} \right|^2 ,$$

and, by the Hadamard determinant theorem,

$$\prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu| \leq n^n \max_{z \in K} |f_1(z)|^2 \cdots \max_{z \in K} |f_{n-1}(z)|^2 .$$

(This inequality is due to Szegö; see Footnote 7 on p. 236 of [3]). Therefore, by Lemma 6,

$$\prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu| \leq 4^{2(n-1)} \cdot n^n .$$

THEOREM 16. *If* Δ_n *is defined by* (10), *then*

$$\Delta_n \leq 2^{4(n-1)} \cdot n^n .$$

Proof. Let $\{z_1, \cdots, z_n\}$ be a system of points with $|z_\mu - z_\nu| \leq 2$ such that

$$\Delta_n = \prod_{\nu=1}^{n} \prod_{\mu \neq \nu} |z_\mu - z_\nu| .$$

Let K be the convex hull of the points z_ν. It follows from Theorem 15 that

$$\Delta_n \leq 2^{4(n-1)} n^n (\text{cap } K)^{n(n-1)} . \tag{12}$$

Since $2 \text{ cap } K \leq \text{diam } K$ and $\text{diam } K \leq 2$, we see that $\text{cap } K \leq 1$. Therefore inequality (12) yields

$$\Delta_n \leq 2^{4(n-1)} n^n .$$

Remark. We can make the following observation in favor of the conjecture of Erdös, Herzog and Piranian about Δ_n. The convex hull K_n of a maximal system $\{z_n^{(n)}, \cdots, z_n^{(n)}\}$ is nearly a disk, for large n. For if this assertion were not true, there would exist a sequence n_k such that the convex sets K_{n_k} converge to a convex set K_0, of diameter at most 2, that is not a disk (for the concept of convergence of convex sets, see for instance [1, p. 34]). Then $2 \text{ cap } K_0 < \text{diam } K_0 \leq 2$, $\text{cap } K_0 < 1$, and $\text{cap } K_{n_k} \leq 1 - \delta < 1$ for k sufficiently large. But then, by inequality (12),

$$\Delta_{n_k} \leq 2^{4(n_k-1)} n_k^{n_k} (1 - \delta)^{n_k(n_k-1)} < 1$$

for large k, contrary to $\Delta_{n_k} \geq n_k^{n_k}$.

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