CONNECTED SETS OF WADA

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1. INTRODUCTION

It is well known that, in the closure of a connected domain D of the euclidean plane E_2, the familiar boring process of Wada [13, pp. 60-62] always gives an indecomposable continuum. Wilder has shown that this is not true in euclidean space E_m ($m \geq 3$); for in [11] he gives constructions which lead to locally connected continua in E_3. We are interested in various types of the Wada tunneling process, and especially in connected towers obtained by using this process a finite or infinite number of times; in fact, our interest was first aroused by the observation that the intersection of some towers of connected Wada domains are indecomposable connected sets with composant properties similar to those of an indecomposable continuum. Various modifications of the Wada tunneling give many types of sets; some of these do not seem now to be characterizable with words so as to give results of generality and interest. However, the Wada construction does lead to some of the more peculiar sets.

Our imbedding space is E_m ($m \geq 3$) or the Hilbert cube I_ω. We obtain indecomposable continua by several methods, all of which depend on modifications of the shielding that Hunter and I used in [2]. The set of perhaps the greatest interest (it is of a new type) we call a *connected set* M *with a set* Z *of indecomposability;* it is a generalization of an indecomposable continuum, since M is indecomposable when it is closed and $\overline{Z} = M$. In general, these new sets, obtained by modifying a netting type of construction used by Wilder in [11], have some properties of composants similar to those of an indecomposable continuum.

All the ideas used here, and most of the proofs, are simple. However it is very difficult to keep this simplicity from being hidden by the complexity of the notation needed for the Wada constructions. Whenever possible, an attempt is made to present an intuitive way of looking at these; this loses something in exactness, but may help to show the simplicity, when accompanied by indications of the more exact construction. The idea of shielding, used in [2], and that of a basic connexe densely extendable over a connected domain, used in [9], are devices whose purpose is to help the intuition; these are both used below. In other places one may find the figures and proofs given by Wilder in [11, pp. 276-278, 290-291] of help. Some of our methods are used, in part, in [2], [7] and [9]; these references may be helpful.

Below, D always denotes a connected domain of our space; in D we construct, by any one of various modifications of the Wada tunneling process, a domain D^1, which may or may not be connected but whose closure $\overline{D}^1$ is connected and is equal to $\overline{D}$. Thus we have a *descending tower* ($D \supset D^1 \supset D^2 \supset \cdots$) of Wada domains; since a tower suggests the upward direction, we put the index h of the h^{th} stage of the tower construction into the upper position: D^h. We denote the boundary of D^h by $F(D^h)$, and we observe that these boundaries give an *ascending tower*

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$\{F(D) \subset F(D^1) \subset F(D^2) \subset \cdots \subset F(D^h) \subset \cdots\}$. The *connected sets of Wada* in which we are interested are either subsets of the union of these $F(D^h)$, or else subsets of the set $\bigcap D^h$.

Fundamental definitions are to be found in [5] and [12]. By a *continuum* we mean a closed and connected set. We denote the null set by $\emptyset$.

Following Wilder [11], we use two fundamental types of construction: we call these a chain type and a netting type. In E_3 the cross section of a tunnel used by Wilder, in either type, is homeomorphic to the interior of a plane circle; in our modifications, the cross section is homeomorphic to the (connected) plane domain bounded by two concentric circles. This difference gives us our desired shielding, and thus leads to connected sets, including continua, with indecomposable properties; Wilder instead gets locally connected continua. In E_m and I_ω, any of these constructions can be justified by the usual theorems on simple chains; but in general we do not give these justifications, since the methods used are well known. We need a common notation for use throughout the various methods of construction: we introduce this in the following section.

2. THE STEP CONSTRUCTION OF A CONNECTED WADA DOMAIN

By a region in E_m we mean any set which is homeomorphic to the interior of an $(m - 1)$-sphere.

We give now the construction of a connected Wada domain D^1, which has the property that $\bar{D}^1 = \bar{D}$. In E_2 this is merely the well-known Wada tunneling process, and there $F(D^1)$ is an indecomposable continuum. Let e_j and d_j be positive numbers such that $\lim e_j = 0 = \lim d_j$, for $j = 1, 2, \cdots$. Let $q_i \in D$ $(i = 1, 2, \cdots)$, and let $\bigcup q_i$ be dense in D. Then there exists a simple chain C_1 of regions $\{R_g\}$ $(g = 1, 2, \cdots, n_1)$ from q_1 to q_2, lying in D and such that each R_g is of diameter less than e_1, and such that $\bigcup R_g$ is connected; furthermore, for each $p \in D - \bigcup R_g$, the distance $\rho(p, \bigcup R_g)$ is less than d_1. Let $\bigcup R_g = A_1^1$. Take the q_i of next highest subscript such that $q_i \in D - \bar{A}_1^1$; to conserve notation, say it is q_3. It is known that the simple chain C_1 can be extended, by a simple chain C_2 of regions $\{R_h\}$ $(h = n_1, n_1 + 1, n_1 + 2, \cdots)$ from q_2 to q_3, so that C_1 together with C_2 is a simple chain from q_1 to q_2. Let $A_2^1 = \bigcup R_h - A_1^1$. The chain C_2 can be taken such that each R_h is of diameter less than e_2, $A_1^1 \cup A_2^1$ is connected and, for $p \in D - (A_1^1 \cup A_2^1)$, the distance $\rho(p, A_1^1 \cup A_2^1)$ is less than d_2. Obviously we can continue this simple chain construction by induction in steps $(C_1, C_2, \cdots)$. We obtain disjoint connected sets A_i^1 $(i = 1, 2, \cdots, n$ and $n = 1, 2, \cdots)$ for each C_i. Each chain region of C_i is of diameter less than e_i, $\bigcup A_i^1$ is connected for each n and, for $p \in D - \bigcup A_i^1$, we have $\rho(p, \bigcup A_i^1) < d_n$. Thus $\bigcup A_n^1$ is the desired connected Wada domain D^1. We say that A_i^1 is the *chain union* for the i^{th} step in the construction of D^1: below, we always put i in the subscript position to indicate this step in the construction of a Wada domain.

We desire intuitive help to explain later constructions, and to see connections between various methods. In [9], and below in Section 5, we obtain this help by means of the *basic connexe densely extendable* over a connected domain D or D^h; this basic connexe is homeomorphic, under an agreement similar to that used in [9], to our desired connected Wada domain D^1 or D^{h+1}. Let D be contained in the xyz-space E_3. Let Q be the bounded connected domain, bounded by $x^2 + y^2 = (1 - z)^2$

and $z = 0$, such that, if $(x, y, z) \in Q$, then $0 < z < 1$. We call Q the *basic conical domain.* In Section 5 we say that the Wada construction, used above to obtain D^1, is a *Wada mapping* of this basic conical domain Q over D, which gives the *Wada image* D^1 of Q; Q is the basic connexe for D^1. It may be an intuitive help to think that the Wada construction stretches Q and places it in position over D densely: it does this in steps, such that, for $i = 1, 2, \cdots$, the part of Q where $z \geq (i - 1)/i$ and $z < i/(i + 1)$ is homeomorphic to A_i^1.

Often the Wada construction is used to give either n disjoint connected domains or infinitely many of these with common boundary. In E_2 this common boundary is an indecomposable continuum: the construction and result are well known. In E_3, Wilder uses this same construction for his Theorem 1 of [11; pp. 275-278]: he obtains a locally connected continuum as this common boundary. Thus this construction gives disjoint connected domains ${}_jD^1$ $(j = 1, 2, \cdots$ or $j = 1, 2, \cdots, n$; $n \geq 2)$ such that $\bigcup {}_jD^1$ is dense in D and the ${}_jD^1$ have common boundary. We let $D^1 = \bigcup {}_jD^1$, and we say that D^1 is a *Wada domain,* although it is not connected. We assign j to its position in ${}_jD^1$ because, in a tower of domains, we have only a minor interest in a disconnected D^1 or D^h; below, j in this position always means that the construction may be for a disconnected D^h; we usually omit j, when $n = 1$.

If we are constructing ${}_jD^1$, the *chain union* becomes ${}_jA_i^1$ in place of A_i^1 above. However, for i held constant, we now let $A_i^1 = \bigcup {}_jA_i^1$ $(j = 1, 2, \cdots$ or $j = 1, 2, \cdots, n)$. The properties of A_i^1 with reference to e_i and d_i are then as they were above.

It is obvious that this construction can be repeated, with the connected Wada domain D^1 or ${}_jD^1$ in place of D above; this gives a connected Wada domain D^2 dense in D^1, or a disconnected Wada domain $D = \bigcup {}_jD^2$, dense in $D^1 = \bigcup {}_jD^1$. The construction can be repeated by induction; thus we shall have connected sets ${}_jA_i^h$ $(h = 1, 2, \cdots)$, of obvious nature, similar to the above for $h = 1$; and we shall have Wada domains D^h (not necessarily connected) such that $D = D^0 \supset D^1 \supset \cdots D^h \supset \cdots$ and D^{h+1} is dense in D^h.

3. THE WILDER CHAIN CONSTRUCTION, (C.1)

Wilder, in his construction for Theorem 1 of [11, pp. 275-278], uses the interior of a semitorus for his chain region; apparently, this is done in order to see easily that the construction is possible. For a Wada domain $D^h = \bigcup {}_jD^h$ $(j = 1, 2, \cdots, n)$, his method of proof needs the following four conditions: (1) Each chain region of ${}_jA_i^h$ is of diameter less than e_i^h, and $\lim e_i^h = 0$ $(i = 1, 2, \cdots)$. (2) For ${}_jK^h = \bigcup {}_jA_f^h$ $(f = 1, 2, \cdots, g)$, each $p \in D^{h-1} - {}_jK^h$ is at a distance from ${}_jK^h$ less than d_g^h, where $\lim d_g^h = 0$ $(g = 1, 2, \cdots)$. (3) No connected rectilinear interval which contains a point of ${}_jA_i^h$ is contained in the union of more than three chain regions of ${}_jD^h$. (4) The boundary set ${}_jF_i^h$, consisting of the part of $F({}_jA_i^h)$ not contained in ${}_jA_k^h$ $(k = i - 1$ or $k = i + 1)$, is arc-wise connected. We call conditions (1) to (4) the *Wilder-Wada Conditions* for the construction of a Wada domain D^h; and we call D^h a *Wilder-Wada domain.* Each chain region is taken homeomorphic to the interior of an $(m - 1)$-sphere, unless otherwise stated. For the connected domain D^h, we omit j from these conditions throughout.

Let $\{D^h\}$ be a tower of Wada domains such that D^1 is dense in $D = D^0$. Let $F^h = F(D^h) - F(D^{h-1})$. We call $\{F(D^h)\}$ a *boundary tower of Wada,* and F^h a *coating* of $F(D^{h-1})$; we say also that F^h is a coating of $F(D^t)$, for all $t < h$. Also, a union of successive coatings, $\bigcup F^e$ ($a \leq e \leq b$; a, b fixed and greater than t) we call a *coating union* of $F(D^t)$. Thus we have:

LEMMA 1. *Let* $\{F(D^h)\}$ *be a boundary tower of Wada, let* $\overline{D} = \overline{D}^h$, *and let each* D^h *be a connected Wada domain. Let* $\{D^g\}$ *be the subclass of* $\{D^h\}$ *such that* D^g *is a Wilder-Wada domain. Then the coating* F^g *of* $F(D^{g-1})$ *is locally and arc-wise connected. If* G *takes on successive values* ($a \leq g \leq b$; a, b *fixed and greater than* t), *then the coating union* $\bigcup F^g$ *of* $F(D^t)$ *is locally and arc-wise connected.*

Proof. We follow Wilder's proof in [11, pp. 277-278]. Let $F_i^g = F(D^g) \cap F(A_i^g)$ and $M' = \bigcup F_i^g$ $(i = 1, 2, \cdots)$. Let $M'' = F^g - M'$. From the nature of F^g we see that $D^{g-1} \supset M''$. We desire to prove that F^g is both locally connected and arc-wise connected.

Consider the case where $p \in M''$. Let R $(R \subset D^{g-1})$ be the interior of an $(m - 1)$-sphere, with p as center and radius e. By the Wilder-Wada Condition (1) and our construction, there exist at most a finite number of the A_i^g with chain regions of diameter greater than $e/4$; let V be the union of these A_i^g. Let R' be the interior of an $(m - 1)$-sphere with p as center, with radius less than $e/4$, and such that $R' \cap V = \emptyset$. Let $q \in R' \cap F^g$, and let pq be a rectilinear interval. Thus $R' \supset pq$. Suppose $x \in D^g \cap pq$. Then there exists an arc axb such that $axb - (a \cup b)$ is contained in $D^g \cap pg$. Thus, by the Wilder-Wada Conditions (1) and (3), axb is contained in the union of at most three chain regions of at most three successive A_i^g. Hence, by the Wilder-Wada Condition (4), there exists an arc ab contained in F^g and in the union of the boundaries of these three regions. We see that each of the three regions is contained in R, since each contains a point of $axb \cap R'$, and each, by the nature of R', is of diameter less than $e/4$. Hence $R \supset ab$.

Let $\{t_f\}$ $(f = 1, 2, \cdots, v$ or $f = 1, 2, \cdots)$ be the class of all possible disjoint arcs $axb \subset pq$; let $\{t_f'\}$ be the class of corresponding arcs ab above. Since $R \supset ab$, $R \supset \bigcup t_f'$. If d_f is the diameter of t_f', we see by the Wilder-Wada Conditions (1), (3), and (4) that $\lim d_f = 0$. Hence $pq - \bigcup t_f + \bigcup t_f'$ is a locally connected continuum, and so it contains an arc $pq \subset R \cap F^g$ for each $q \in R' \cap F^g$. Therefore F^g is connected *im Kleinen* at p and, by Theorem 7 of [5, p. 94], it is therefore locally connected at p; in fact, the proof shows that F^g is locally arc-wise connected at p. The case where $p \in M'$ can be treated in a similar manner. It then follows that F^g is arc-wise connected, because by the local property, there exist arcs pa and bq, for $p, q \in F^g$; and there exists an arc ab contained in a finite number of the F_i^g, by the Wilder-Wada Condition (4). Therefore the coating F^g of $F(D^{g-1})$ has the desired properties of the lemma.

Consider now the case where D^g is a Wilder-Wada domain for all g ($a \leq g \leq b$; a, b fixed and greater than t). We wish to show that the coating union $\bigcup F^g$ of $F(D^t)$ is both locally connected and arc-wise connected. If in the proof above we use D^a in place of D^{g-1}, it follows at once that $\bigcup F^g$ is locally arc-wise connected.

We now prove that $\bigcup F^g$ is arc-wise connected. Let $p, q \in \bigcup F^g$. Then there exist u and v such that $p \in F^u$, $q \in F^v$, and $a \leq u \leq w \leq v \leq b$. Hence $\bigcup A_i^v$ is dense in both F^u and F^v, and $\bigcup F^w$ is locally arc-wise connected. Thus there exists p' in some F_i^v, and so there exists an arc $pp' \subset \bigcup F^w$. Similarly, there exists q' in some F_t^v and an arc qq' contained in $\bigcup F^w$. By the Wilder-Wada Condition (4), there exists an arc $p'q' \subset F^v$ from F_i^v to F_t^v. Hence we see that $\bigcup F^w$ is arc-wise connected, and the lemma is true.

COROLLARY 1.1. *If* $\{F(D^h)\}$ *is a boundary tower of Wada,* $\overline{D} = \overline{D}^h$, *and each* D^h *is a connected Wilder-Wada domain, then* $\bigcup F(D^h) - F(D)$ *is arc-wise connected; it is locally connected for* $h = 1, 2, \cdots, n$; $n = 1, 2, \cdots$.

Problems of interest concern the types of sets obtained when the Wilder-Wada Conditions are changed. If (4) is changed to

(4') the closure of ${}_jF_i^h$ is a compact continuum,

then we retain local connectedness in Lemma 1, although we may lose arc-wise connectedness. If we did not take $\lim e_i^h = 0$ in (1), then an $F(D^f)$ could be the cartesian product of an indecomposable continuum and the unit interval. In Section (4), we obtain indecomposable continua, in spite of the fact that the Wilder-Wada Conditions (1) to (4) may be satisfied: this results because our chain regions are annular.

4. AN INDECOMPOSABLE CONTINUUM CHAIN CONSTRUCTION, (C.2)

The older examples of indecomposable continua each had shielding (although it was not called by that name); it was by means of this that indecomposability was established. We define shielding as follows:

Let W be a connected subset of E_m. Let $\{P(R', R'')\}$ be the class of all pairs $P(R', R'')$ of regions such that $\overline{R}' \cap \overline{R}'' = \emptyset$ and $R' \cap W \neq \emptyset \neq R'' \cap W$. Let $\{T_i\}$ be a class of subcontinua of E_m. Let there exist, for each pair $P(R', R'')$, a T_i and a domain H such that $F(H) \subset T_i \cup F(R')$ and both H and $E_m - H$ have points in common with $W \cap R''$, and such that $(F(H) - F(R')) \cap W = \emptyset$. Let M be any set such that $M \cap W = \emptyset$ and $M \supset \bigcup T_i$. Then we say that M, or $\{T_i\}$, *shields* W *densely* and is a *shielding* of W. If T_i is an arc or a simple chain, each of which has meaning for $<$ with respect to its elements, we say that T_i passes from R' to R'', then *back again* through R'.

Let I be an indecomposable continuum in E_2; let $\{R_k\}$ be a countable basis of regions for I, and let I be such that, for every R_k and R_j $(k \neq j)$, there exists an arc t from R_k to R_j to R_k, with $t \subset S - I$. The class of arcs t is a class $\{T_i\}$ of shielding of I.

LEMMA 2. *Let the class* $\{T_i\}$ *of disjoint connected sets be such that* $\bigcup T_i$ *shields densely the connected set* W. *Then* W *is an indecomposable connected set.*

Proof. Suppose that there exist connected sets U and V such that $W = U \cup V$ and $\overline{U} \neq \overline{W} \neq \overline{V}$. Then there exists a pair $P(R', R'')$ of regions such that

$$\overline{R}' \cap \overline{V} = \emptyset = \overline{R}'' \cap \overline{U} \quad \text{and} \quad R' \cap U \neq \emptyset \neq R'' \cap V.$$

Hence there exists a continuum $F(H)$, by the above, such that $F(H) \cap V = \emptyset$, although $H \cap V \neq \emptyset \neq (E_m - H) \cap V$. Thus V is the union of mutually separated sets $V \cap H$ and $(E_m - H) \cap V$, and so V cannot be connected. Hence W is indecomposable.

We need the following here and later. Let $p, q \in D \subset E_m$. Let W be the union of the chain regions of a simple chain C from p to q, whose last link is R_n. Then we say that $F(W) - F(R_n) = Z$ is a *cylindrical surface open at one end.* Let S' and S'' be the interiors of two $(m - 1)$-spheres such that $S' \supset \overline{S}''$. We say that $S = [p \mid p \in S'$ and $p \notin \overline{S}'']$ is an *annular region* of E_m. We call S' the *outer* and S'' the *inner region* of S. Take pairs $P(S', S'')$ and a simple chain of inner regions S_i''

(i = 1, 2, ⋯) from p to q such that the outer regions S'_i also constitute a simple chain from p to q: we call this a *simple chain* C *of annular regions from* p to q.

Let K' be the union of the outer regions of C, and K" the union of the inner regions; and let $K = K' - \overline{K}''$. Then, by an *annular cylindrical domain open at one end,* we mean the set of all points of K except those in the closure of the last annular link of C.

We now give an example of an indecomposable connected set $M \subset D \subset E_2$, due to E. W. Miller [7, Example B, p. 796]. We then show how to modify this to obtain an indecomposable connected set $M \subset D \subset E_m$; in turn, we modify this to obtain an indecomposable continuum.

Let A be a simply connected domain such that $\overline{A} \subset D \subset E_2$. Let $\{R_f\}$ be a countable basis of regions of $D - \overline{A}$. Let $\{z_i\}$ (i = 1, 2, ⋯) be a class of disjoint simple continuous arcs, such that each z_i is contained in D - A and $z_i \cap F(A)$ is an end point of z_i; further, let there exist, for each pair P(R', R") of regions of $\{R_f\}$, a z_i passing from R' to R", then back again through R'. Let $M = (\overline{D} - A - \bigcup z_i)$. Then M is connected. Since $\bigcup z_i$ shields M, by Lemma 2, M is indecomposable.

Let now $D \subset E_m$, and let A and $\{R_f\}$ be as above for E_m. Let $\{Z_i\}$ (i = 1, 2, ⋯) be a class of disjoint cylindrical surfaces, each open at one end; let also the first region in the simple chain, which gives Z_i, have points common with A. For each pair P(R', R") of regions of $\{R_f\}$, let there exist a Z_i whose construction chain has a subchain, with end regions contained in R' and with a mid-link contained in R". Let $M = (\overline{D} - A - \bigcup Z_i)$. Then, as above, M is an indecomposable connected set. Now let each Z_i be an annular cylindrical domain open at one end. Then the same argument shows that M is an indecomposable continuum. The construction presents no difficulty. In order for the notation to agree with that of Section 2, let $A_1^1 = A \cup Z_1$ and $A_i^1 = Z_i - A$ (i = 2, 3, ⋯). Then $D^1 = \bigcup A_i^1$ (i = 1, 2, ⋯), as in Section 2. Similarly, ${}_jA_i^h$ is defined as in Section 2. We note that the construction of the class $\{A_i^1\}$ is by induction, as in Section 2: at each step in this process, a P(R', R") is used to obtain the next A_i^1. The pair P(R', R") for A_i^1 must be chosen so that no A_j^1 $(j < i)$ contains either R' or R"; thus here we do not use all pairs P(R', R"), although for the two other cases above we can do so. We note also that $M = F(D^1)$. Therefore we have

LEMMA 3. *Under the construction* (C.2), *there exists a tower of connected Wada domains* D^h (h = 1, 2, ⋯, v *or* h = 1, 2, ⋯) *such that each* $F(D^h)$ *is an indecomposable continuum and* $\overline{D}_h = \overline{D}$; *also, each coating* F^h *of* $F(D^t)$ $(t < h)$ *and each coating union is an indecomposable connected set.*

We delay giving theorems on towers, where the types of constructions are mixed, until we have other types. We consider next chain constructions from the standpoint of Wada mappings of basic connexes.

5. BASIC SETS DENSELY EXTENDABLE OVER D BY A WADA CONSTRUCTION

We consider here two basic connexes and their possible shielding, which can be densely extended over D by a Wada construction, as suggested in Section 2. We say that this Wada construction gives a *Wada mapping* of the basic connexe, together with the possible shielding, when these are stretched and placed into position over D densely, as suggested for Q in Section 2. We call the set of points as extended over D the *Wada image* of the basic set.

Let the imbedding space be the xyz-coordinate space. The line interval from (0, 0, 0) to (0, 0, 1), but minus (0, 0, 1), we call the *basic ray* N. We note that N is the axis of the *basic conical domain* Q of Section 2. Hunter and I in Theorem 1 of [2, pp. 4-5] mapped N, with its shielding, over D, and obtained a Wada image, which was an indecomposable connected set. See [9, pp. 817-818] for another explanation of this and related Wada images. The Wada image, for Theorem 1 of [2], was $\bigcup t_i$ $(i = 1, 2, \cdots)$, where t_i is an arc $p_{i-1} p_i$, and each $\bigcup t_f$ $(f = 1, 2, \cdots, 3$ and $e = 1, 2, \cdots)$ is an arc from p_0 to p_e. This basic ray also was mapped, in [2, pp. 7-8], so that its image was a decomposable connected set in Theorem 5, and a locally connected one in Theorem 6; this latter result has similarities with Wilder's mapping of Q in Theorem 1 of [11], described in Sections 2 and 3 above.

We now take N, with shielding, and obtain an indecomposable continuum construction (C.2); we then take Q in place of N, and obtain the same result.

Let us consider the nature of the inverse image of the shielding, for the indecomposable trajectory of Theorem 1 of [2]. Let the conical surface Z_i'' $(i = 1, 2, \cdots)$, with vertex at (0, 0, 1), intersect z = 0 in $x^2 + y^2 = (1/i)^2$. Let Z_i' be the part of this lying between z = 0 and $z = i/(1 + i)$. The basic ray N is a limiting set of the class $\{Z_i'\}$; we say that the Z_i' *close down* on N. Each Z_i' has as its Wada image the cylindrical shielding Z_i in [2, p. 5]; as i increases, the stretching by the Wada mapping becomes greater, as suggested for Q in Section 2. Thus $N \cup (\bigcup Z_i')$ is mapped, in the proof of Theorem 1 of [2], into $(\bigcup t_i) \cup (\bigcup Z_i)$, where $\bigcup t_i$ is an indecomposable connected set, and $\bigcup Z_i$ is its shielding.

Let now A' be the domain in xyz-space bounded by z = 0, z = -1, and $x^2 + y^2 = 1$. Then, in $A' \cup N \cup (\bigcup Z_i')$, each Z_i' above is a cylindrical surface, open at one end, as defined in Section 4. Change each Z_i' into an annular cylindrical domain, open at one end, as defined in Section 4, where these close down on N in a manner similar to that of the surfaces. Now map $A' \cup N \cup (\bigcup Z_i')$ into its Wada image in D such that: A' goes into A, where A is a connected domain and $\bar{A} \subset D$; each Z_i' maps into Z_i; if $D^1 = A \cup (\bigcup Z_i)$, then $\bar{D}^1 = \bar{D}$. By Lemma 2 and by Knaster and Kuratowski's Theorem 37 [4], we see that $F(D^1)$ is an indecomposable continuum, when each Z_i' is an annular domain; also, D^1 is a connected Wada domain, since Z_i is here a domain. If Z_i' is a cylindrical surface, then $F(D^1) - \bigcup Z_i$ is an indecomposable connected set. In the first case, the image $\bigcup t_i$ of N is a composant of $F(D^1)$.

Substitute now the basic conical domain Q in place of N above, and let $\{Z_i'\}$ close down on $\bar{Q}$ rather than on N. Let

$$D^1 = A \cup (\bigcup Z_i) \cup (\text{image of } Q).$$

If Z_i' is an annular domain, then $F(D^1)$ is an indecomposable continuum. Although, both here, and for Wilder's Theorem 1 in Section 3, the basic connexe mapped by the Wada construction is Q, in one case indecomposability is obtained, while in the other local connectedness is obtained. The presence of shielding causes the difference.

The method of this Section has been intuitive: its purpose is to give quick understanding of the possible (C.1) and (C.2) types of construction. These are both chain types. Our remaining constructions are netting types.

6. THE WILDER NETWORK CONSTRUCTION

We are interested in two network constructions in D, each of which gives a connected Wada domain D^1 such that $\overline{D}^1 = \overline{D}$. One of these is that used by Wilder, in his proof of Theorem 8 of [11, pp. 290-291], and with its use $F(D^1)$ is a locally connected continuum. With the use of the other, $F(D^1)$ is a continuum with a set Z of indecomposability; when $Z = F(D^1)$, $F(D^1)$ is an indecomposable continuum. The second network construction (C.4) is a modification of the first, which is (C.3) below.

We describe the constructions (C.3) and (C.4) in the xyz-space E_3. Each of them uses a chain construction of nettings, and this extends to higher-dimensional spaces. In (C.4), chains of annular regions, which we call *annular chains*, are used. See Section 4 for these.

Through each point (x, y, z), where x, y, and z are integers, take straight lines parallel to the three coordinate axes, respectively. This fills E_3 with a set of unit cubes, with edges on these lines and vertices at these points. Any set, homeomorphic to the union of these lines, we call a *netting of arcs;* we call the homeomorph of one of these cubes a *netting cube.* Consider now the connected domain H of E_3. Omit from the above netting an edge of a netting cube whenever the edge is not contained entirely in H; if the remaining set is connected, and if each edge that remains is an edge of a cube contained in H, and H contains at least one netting cube, we say that this set is a *netting of* H. Let $d_i > 0$, and let N be a netting of H. We say that the *netting* N d_i-*fills* H if every region R in H, of diameter greater than d_i, contains a netting cube of N (we recall that "region" always denotes the interior of a sphere). We use N to denote both the netting and the point set composing that netting.

Let N be a netting of arcs of a connected domain H. Let ab be any simple continuous arc of N; replace ab by a simple chain of regions, and let C_{ab} be the union of its chain regions. If t and t' are any two arcs of N, with vertex p of N such that $p = t \cap t'$, then $C_t \cap C_{t'}$ is the chain region, with p as center, common to C_t and $C_{t'}$; if t and t' have an arc in common, then the corresponding chains have a simple chain in common. With this replacement of an arc of N by a simple chain, we call the resulting set a *simple chain netting* N'. If N is a netting of H, then we take $N' \subset H$. A *chain netting cube* of N' is the subset of N' obtained by replacing the edges of an N cube by the simple chains of N'.

Let N' be a simple chain netting of the domain H. Replace the simple chains of N' by annular chains: then the inner regions of these chains form a simple chain netting N_1, and the outer regions also form a simple chain netting N_2. Then we say that $N = N_2 - \overline{N}_1$ is an *annular chain netting.* It has the property that $H - \overline{N}$ is the union of two disjoint connected domains; one of these is N_1, and it is called the *inner domain* of N. Let $p \in N$, and let R be a region such that $p \in R$, such that R contains a point of the inner domain of N, and also of the outer domain, and such that the set $Q = N \cap R$ is homeomorphic to the interior of a sphere. When these conditions are satisfied, we say that Q is an *opening* into N. (In order to see that there exists a netting N with opening Q, and with a connection T described below, one may take the basic netting of arcs with cube edges which are rectilinear intervals. We need this existence below, although we do not wish to confine our nettings of arcs in this manner.)

Let N and N' be disjoint annular chain nettings. Let T be a simple chain of regions with the following properties: T has only points of its first region in common with N, and only points of its last region in common with N'; T does not have a point in common with either the inner domain of N or the inner domain of N'; each link of T is homeomorphic to the interior of a sphere. Then T is said to be a *connection* from N to N'. We define in a similar manner a connection between N and N' when each of these is a simple chain netting.

In both of the constructions (C.3) and (C.4), we construct a class of disjoint nettings $\{N_i\}$ $(i = 1, 2, \cdots)$ such that the maximal diameter of a chain netting cube of N_i is less than e_i and $\lim e_i = 0$. It follows that each chain region of N_i is of diameter less than e_i. Let $\{T_i\}$ be a class of disjoint connections such that T_i is a connection from N_i to N_{i+1}, and the diameter of T_i is less than e_i. Let $\{Q_i\}$ be a class of openings such that Q_i is an opening of N_i and $Q_i \cap T_i = \emptyset$, in case N_i is an annular chain netting. Let $d_i > 0$, and let $\lim d_i = 0$. We assume that $\bigcup (N_i \cup T_i)$ $(i = 1, 2, \cdots)$ is contained in D.

In the construction (C.3), each N_i is a simple chain netting. For $i = 1$, let $V_i = \emptyset$; for $i > 1$, let $V_i = \bigcup (N_f \cup T_f)$ $(f = 1, 2, \cdots, i - 1)$. Then the classes $\{N_i\}$ and $\{T_i\}$ are taken so that $N_i \cup T_i$ d_i-fills $D - V_i$. Then we say that $\bigcup (N_i \cup T_i)$ $(i = 1, 2, \cdots)$ is the *network of the* (C.3) *construction;* here it is the domain D^1; if one substitutes D^1 for D in the definition of the network, then it is D^2.

In the construction (C.4), each N_i is an annular chain netting and each N_i has an opening Q_i. In the definition of the (C.3) network, replace N_i by $N_i - \overline{Q}_i$. Then we say that $\bigcup (T_i \cup N_i - Q_i)$ is the *network of the* (C.4) *construction.* Since it is the network for D, it is D^1. Below, we place a further condition upon the (C.4) network construction.

In each case, the network for D is a Wada connected domain D^1.

7. THE WILDER NETWORK CONSTRUCTION (C.3)

The proof, given by Wilder for his Theorem 8 in [11, pp. 290-291], is for a disconnected Wada domain $D^1 = {}_1D^1 \cup {}_2D^1$; also, his class $\{T_i\}$ of connections is more complicated, and this implies that his ${}_jD^1$ is uniformly locally connected; however, the uniformity depends on the nature of F(D). In a tower, $F(D^{h-1})$ would not permit this uniformity for D^h. We need a netting type of construction, so that $F(D^h)$ is a locally connected continuum, and our (C.3) network construction can give this. For if we take this construction so that the Wilder-Wada Conditions of Section 3 are satisfied, the proof of Lemma 1 will hold, in the case when the D^h of the tower are of (C.3) construction.

In the (C.3) network of the previous section, let $A_1^1 = N_1 \cup T_1$ and, for $i > 1$, let $A_i^1 = N_i \cup T_i - A_{i-1}^1$. Then $D^1 = \bigcup A_i^1$ $(i = 1, 2, \cdots)$. We construct $A_1^1, A_2^1, A_3^1, \cdots$ in that order, satisfying the conditions of Section 6. When D^1 is not connected, we define ${}_jA_i^1$ in a manner similar to $A_i^1 = {}_1A_i^1$. Then $D^1 = \bigcup_j D^1$, as in Sections 2 and 3; we define ${}_jA_i^h$ and D^h, and obtain a tower of Wada domains here, as we did there.

Our construction conditions in Section 6 insure that we have the Wilder-Wada Conditions (1), (2) and (4); it is to be noted, in Condition (3), that the 'three' could be any greater fixed finite number: this makes it easier to see that the (C.3) network can be assumed to satisfy all of these conditions. Thus we have:

COROLLARY 1.2. *If* $\{D^h\}$ *is a tower of Wada of* (C.3) *construction,* $\overline{D} = \overline{D}^h$, *and the Wilder-Wada Conditions hold for each* D^h, *then* $\bigcup F(D^h) - F(D)$ *is arc-wise connected; and it is locally connected, for* $h = 1, 2, \cdots, n$ *and* $n = 1, 2, \cdots$.

8. THE (C.4) NETWORK CONSTRUCTION

The (C.4) network of Section 6 has associated with it a connected domain D^{h-1}, where D is D^0. We also associate with it a set $Z' \subset D$ and a countable basis $\{H_i'\}$ of regions of D about Z'. The set Z' may be any subset of D. Each H_i' is homeomorphic to the interior of an (m - 1)-sphere. If $z \in Z'$, R is any region in D, and $z \in R$, then there exists an H_i' such that $\overline{H}_i' \subset R$.

The construction of the (C.4) network is by induction. Thus one takes $(N_1, N_2, N_3, \cdots, N_i, \cdots)$, in the order of the subscripts, and with the properties stated in Section 6; with each N_i is associated its opening Q_i and its connection T_i. The first steps in this construction are as follows. Let $H_2 = H_1' \in \{H_i'\}$. Construct N_1, and take $Q_1 \cup T_1 \subset H_2$. Then take N_2 so that it d_2-fills $D - (N_1 \cup T_1 - Q_1)$ and has the other properties desired in Section 6, also so that $H_2 \supset Q_2 \cup T_2$. Let H_4 be the H_i' $(i \neq 1)$ of smallest subscript such that $(N_1 \cup T_1) \cup (N_2 \cup T_2)$ does not contain H_4. Next, take a subclass $\{H_{2f}\}$ of $\{H_i'\}$ such that H_{2f} contains the openings into both N_{2f} and N_{2f-1}, and such that, H_{2f} contains the connections from N_{2f} to both N_{2f-1} and N_{2f+1}; that is, let

$$H_{2f} \supset (Q_{2f-1} \cup T_{2f-1}) \cup (Q_{2f} \cup T_{2f}).$$

In the inductive construction, H_{2f} is always taken as the next H_i' which is not contained in the union of the $H_t \cup T_t$ previously constructed.

Hence, for a given N_{2f}, the (C.4) Network has these properties: If $i < 2f$, then the inner domain of N_{2f} does not contain a point of N_i. If $i > 2f$, there exist p and q in N_i such that p is a point of the inner domain of N_{2f} and q is in its outer domain; there exists an arc $pq \subset N_i$; and the arc pq must go through H_{2f}, as must every arc of N_i. For $i > 2f$, we say that N_i has the property of *passing through* H_{2f} in going from the inner to the outer domain of N_{2f}, since its arcs and chains have this property.

We can let $A_1^1 = (N_1 - \overline{Q}_1) \cup T_1$ and, for $i > 1$, we write

$$A_i^1 = (N_i - \overline{Q}_i) \cup T_i - A_{i-1}^1.$$

This is similar to what we did in Section 7, and it brings the notation into agreement with that used in Sections 2 and 3. In an obvious manner, we give meaning to ${}_jA_i^h$ and D^h. Thus we have

LEMMA 4. *There exists a tower* $\{D^h\}$ *of connected Wada domains, where each* D^h *is obtained by the network construction* (C.4), *and* $\overline{D^h} = \overline{D}$, *for each* h; *each* $F(D^h)$ *is an indecomposable continuum.*

Proof. Let $Z' = D$. We see that $F(D^h)$ is a continuum. Suppose it is decomposable into proper subcontinua V and W. Then a pair $P(R', R'')$ of regions can be taken such that

$$\overline{V} \cap \overline{R}'' = \emptyset = \overline{W} \cap \overline{R}' \quad \text{and} \quad V \cap R' \neq \emptyset \neq W \cap R''.$$

Because of the nature of $\{H_i'\}$, there must exist an $H_{2f} \subset R'$. For large enough $i > 2f$, there exist points of $N_i \cap R''$ (and therefore points of $R'' \cap F(D^h)$) in both the inner and outer domain of N_{2f}. Since $N_{2f} \cap F(D^h) = \emptyset$, we see that $\bigcup N_{2f}$ gives shielding to $F(D^h)$: obviously, the N_i here are those used in the construction of D^h. By the proof of Lemma 2, it follows that $F(D^h)$ is an indecomposable continuum.

9. THE (C.4) NETWORK AS A CONTINUUM WITH A SET OF INDECOMPOSABILITY

In the previous section, a set of regions about Z' contained all the netting openings and connections. If $Z' \neq D$, then $\bigcup A^1_{2f}$ now does not shield $F(D^1)$ densely in D; instead, its shielding for $F(D^1)$ is certain only for Z'. Thus we need to modify our definition of shielding, which we do as follows.

Let Z and W be nondisjoint subsets of E_m, and let W be connected. Let $\{T_i\}$ be a class of connected subsets of $E_m - W$. Let $P(R', R'')$ be any pair of regions such that $R'' \cap W \neq \emptyset \neq Z \cap (R' \cap W)$. Then we say that $\{T_i\}$, or any set M which contains $\bigcup T_i$, *shields* W *out from* Z, or is a *shielding of* W *out from* Z, if and only if, for every pair $P(R', R'')$, there exists a domain H such that $F(H)$ is a continuum contained in some $T_i \cup F(R')$, such that $F(H) \cap (W - \overline{R'}) = \emptyset$, and such that $R'' \cap W$ contains points both of H and of $E_m - H$; in other words, if $F(H)$ is a separating boundary continuum exterior to $\overline{R}'$ which separates W in R''.

We need to define the following concepts. The subset V of the connected set W is a *region-containing subset* of W, if for some region R it is true that $V \supset R \cap W$ and $R \cap W \neq \emptyset$. We say that a nonnull subset Y of W is *a set of indecomposability* of W if and only if, for every region-containing subset C of W, $\overline{C} \supset Y$; then, Z is *the set of indecomposability* of W if Z is the maximal Y. We say that W is a *connected set, with a set* Z *of indecomposability,* if a nonnull set Z of indecomposability exists. If W also is closed, we say that W is a *continuum with a set* Z *of indecomposability.*

If I and I' are two indecomposable continua in E_2, with a set Z of tangency, then $I \cup I'$ is a continuum with the set Z of indecomposability. Also the closure of the biconnected set S, with dispersion point a of [4: p. 241], is a continuum with set $a = Z$ of indecomposability.

LEMMA 5. *Let* D^1 *be a connected Wada domain of* (C.4) *network construction, such that* $\overline{D}^1 = \overline{D}$ *and* $D^1 = \bigcup A^1_i$. *Let also* $\{A^1_{2f}\}$ *shield* $F(D^1)$ *out from a set* Z. *Then every region-containing connected subset* V *of* $F(D^1)$ *is dense in* $Z \cap F(D^1)$.

Proof. Suppose the lemma is false, so that there exists a region-containing V of $F(D^1)$ such that $\overline{V}$ does not contain $Z \cap F(D^1)$. Let R' be a region, and z a point in Z such that $z \in R'$ and $\overline{R}' \cap \overline{V} = \emptyset$. Since V is region-containing, there exists a region R'' such that $\overline{R}' \cap \overline{R}'' = \emptyset$ and $\overline{V} \supset R'' \cap F(D^1)$. Recall now the class $\{H_{2f}\}$ of the (C.4) construction of Section 8. There exists an H_{2f} and the corresponding A^1_{2f} with the following properties: R' contains H_{2f}; A^1_{2f} contains points of $R'' \cap \overline{V}$; H_{2f} contains the opening of A^1_{2f}; and H_{2f} contains the connections from A^1_{2f} to A^1_t $(t = 2f - 1$ and $t = 2f + 1)$. Since $D^1 \supset A^1_{2f}$, and so $A^1_{2f} \cap \overline{V} = \emptyset$, it follows that $A^1_{2f} \cup F(R')$ contains a separating boundary continuum $F(H)$ of the definition of shielding out from Z. Thus V cannot be connected, which is a contradiction. Therefore the lemma is true.

THEOREM 1. *There exists a connected Wada domain* D^1 *of the* (C.4) *network construction, such that* $\overline{D}^1 = \overline{D}$, $F(D^1)$ *is a continuum with a set* Z *of indecomposability, and* $F(D^1)$ *is not an indecomposable continuum.*

Proof. Here $D^1 = \bigcup A^1_i$, and $A^1_i = {}_1A^1_i$ satisfies the Wilder-Wada Conditions (1) and (2). Thus D^1 is a connected Wada domain. For any chosen subset Z' of D, the class $\{H_{2f}\}$ has a subsequence closing down on each $z \in \overline{Z}' \cap F(D^1) = Z$. Each H_{2f} contains the opening and the connection of both A^1_{2f-1} and A^1_{2f}. For $t > 2f$, each A^1_t, passes through the opening of A^1_{2f} and has points in common with both the inner and the outer domain of N_{2f} of A^1_{2f}. Therefore it follows that $\{A^1_{2f}\}$ is a shielding of $F(D^1)$ out from Z, and therefore the hypothesis of Lemma 5 is satisfied. Therefore $F(D^1)$ is a continuum with the set Z of indecomposability.

We show now that D^1 can be taken so that $F(D^1)$ is a decomposable continuum. Let R be a region, and Z' a subset of D such that $\overline{R} \cap \overline{Z'} = \emptyset$. We take the (C.4) network D^1 as in Section 8. However, consider the nature of the class $\{N_i\}$ of netting in Section 6. The opening and connection of each N_i is in a region H'_t about a point of Z'. Hence this opening and connection can be taken with no points common with R. Because of its netting construction, $N_i - R$ can be taken connected. Then $\bigcup (N_i - R) = D^1 - R$ is connected. Thus $F(D^1) - R$ can be taken connected. An indecomposable continuum cannot have this property, and so $F(D^1)$ can be decomposable.

Obviously, we can use nettings of cylindrical surfaces, each with opening and connection, in place of nettings of annular chains in (C.4). Then $D - \bigcup A^1_i$ and $\overline{D} - \bigcup A^1_i$ would each be indecomposable connected sets; if we define the term "closed" for these sets as "closed in itself," they are examples of indecomposable continua, which are not bicompact.

10. CONTINUA WITH SETS OF INDECOMPOSABILITY

We now give theorems which show relations between continua and sets of indecomposability; especially, we develop the composant theory for them.

THEOREM 2. *If* M *is a continuum with a set* Z *of indecomposability containing at least two points, then* M *is non-aposyndetic at each point of* Z, *and it may be aposyndetic elsewhere.*

Proof. Consider the case where p and q are in Z and M is aposyndetic at p, with respect to q. By definition [3, p. 404], there exists a subcontinuum H of M, and an open subset U of M, such that $M - q \supset H \supset U \supset p$. By definition, H is region-containing, and so $H \supset Z$, since Z is a set of indecomposability of M. But then $M - q \supset H \supset Z \supset q$, which is a contradiction. Therefore M is non-aposyndetic at each point of Z. If $p \in M - Z$, we conclude from the last part of the proof of Theorem 1 that M could be aposyndetic at p.

THEOREM 3. *There exists a connected Wada domain* D^1, *of* (C.4) *construction, such that* $F(D^1)$ *is a continuum, whose set of indecomposability consists of a single point* Z, *and such that* $F(D^1)$ *is locally connected at* Z.

Proof. The regions of the class $\{H'_i\}$ of Section 8 close down on the point Z. We note that the netting N_1 can be taken so that each $N_1 \cap H'_i$ is connected; similarly, each $N_t \cap H'_i$ $(t > 1)$ can be taken connected. Thus $F(D^1)$ is locally connected at Z.

In Theorems 4 and 12 we need the following definitions, which we gave in [8, p. 90]. Let the connected set M be the union of the sets M_i $(i = 1, 2, \cdots, n)$, and let M' be any one of these. By the essential part of M', always denoted by E(M'), we mean the part of M' which is not contained in the union of the closures of the remaining sets M_i. If each $E(M_i) \neq \emptyset$, then we say that M is the *essential union* of these M_i. We say that M has R_x*-local essential union at* x if and only if, for every region R_x containing x, there exists a region $R'_x \subset R_x$ such that $R'_x \cap M$ has essential union of sets $R'_x \cap M_i$ $(i = 1, 2, \cdots, n)$, for $M_i \subset M$. Further, M is an n-*indecomposable connected set* if and only if M has essential union of the connected subsets M_i (i = 1, 2, ⋯, h) for h = n but not for h = n + 1. And we say that M is *locally* n-*indecomposable at* x if and only if it is connected and if, for every R_x, it has R_x-local essential union for connected subsets M_i (i = 1, 2, ⋯, h) for h = n, but not for h = n + 1. However, for $n = \aleph_0$, we change "not for h = n + 1" in both cases to "not for a higher power h." Thus we have

THEOREM 4. *If* M *is a continuum whose set* Z *of indecomposability* Z *is the closure of a domain with respect to* M, *then* M *is locally* 1-*indecomposable at each point of this domain, but it may be locally* $\aleph_0$-*indecomposable elsewhere.*

The proof follows easily from our definitions and from the construction of D^1 in Theorem 1.

Let M be a continuum with the set Z of indecomposability, and let p, q ∈ M; by T_{pq} we mean a subcontinuum of M which contains $p \cup q$. We define the *composant* T_p *of* M *with respect to* p as follows:

$$T_p = [q \mid \text{there exists a non-region-containing } T_{pq}].$$

A composant of an indecomposable continuum M is an example.

Notation. Below, M is a compact continuum with the set Z of indecomposability. Let $\{H_i''\}$ be a class of regions with respect to M such that for each i, $H_i'' \cap Z \neq \emptyset$, and such that the class has, for each z ∈ Z, a subsequence of regions closing down on z. Then, for each H_i'' and each $p \in M - \overline{H}_i''$, there exists a maximal subcontinuum W_i of $M - H_i''$ such that $p \in W_i$. We call this the W_i for $p \notin H_i''$; and by $\{W_i\}_p$ and $(\bigcup W_i)_p$ we denote the classes of these subcontinua and their union, respectively. If $T_p = (\bigcup W_i)_p$, then T_p is said to be a *composant of countable union type.* See M of Example α_2, which follows the proof of Theorem 6 below.

THEOREM 5. *Let* M *be a compact continuum with the set* Z *of indecomposability, and let* p ∈ M *and* Z ≠ p. *Then* $T_p \supset (\bigcup W_i)_p \neq \emptyset$, *and* T_p *is dense in* Z.

Proof. Since M is a compact continuum, there exists (by Theorem 34 of [5, p. 21]) an irreducible continuum joining p and $\overline{H}_i''$; let W_i be the maximal subcontinuum of $M - H_i''$ that contains p. Since there exists $z \in H_i'' \cap Z$ and $H_i'' \cap W_i = \emptyset$, the subcontinuum W_i does not contain Z. Thus, by definition of Z, W_i is not a region-containing subcontinuum of M. Hence the composant T_p contains $(\bigcup W_i)_p$. We see that $(\bigcup W_i)_p \neq \emptyset$, because Z ≠ p, and so there exist an H_i'' such that $p \notin \overline{H}_i''$.

If $p \notin Z$, then there exists a W_i for each H_i'', and a subsequence of $\{H_i''\}$ closes down on each z ∈ Z. Hence T_p contains the set $(\bigcup W_i)_p$, which has z as a limit point. Therefore T_p is dense in Z when $p \notin Z$. Consider the remaining case, where p ∈ Z ≠ p. Here also there exists a subsequence of $\{H_i''\}$ closing down on each $z' \in Z - p$, and so $(\bigcup W_i)_p$, contained in T_p, has z' as a limit point. Hence the theorem is true.

THEOREM 6. *If* M *is a compact continuum with the set* Z *of indecomposability, and* Z *contains a proper open subset of* M, *then every* T_p *is a composant of countable union type. If* Z *does not contain a proper open subset of* M, *then there can exist a composant* T_p *which is not of this type.*

Proof. If Z contains a proper open subset of M, then any subcontinuum W' containing Z contains a region with respect to M, and therefore W' is region-containing. Let W be a non-region-containing subcontinuum of M such that p ∈ W. Then, by the nature of W', W does not contain Z, and so there exists $z \in Z - W$. Hence there exists an H_i'' such that $H_i'' \cap W = \emptyset$ and $z \in H_i''$. Thus there exists a W_i which contains W. Therefore $T_p = (\bigcup W_i)_p$, and so T_p is of countable union type.

Let M be contained in the xyz-space, and let Z be a point on the z-axis. Let C be the Cantor ternary set on the rectilinear interval from (1, 0, 0) to (0, 1, 0). For $c \in C$, let P_c be the plane containing the z-axis and c. Let I_c be an indecomposable continuum such that $I_c \subset P_c$ and $Z \in I_c$. For $c \neq e$, I_c may be revolved about the z-axis into I_e; $I_c - Z$ and $I_e - Z$ are disjoint. The set $\bigcup I_c = M$ is a continuum with the set Z of indecomposability. The regions of $\{H_i''\}$ close down on the point Z. Hence there does not exist a maximal subcontinuum W_i of $M - H_i''$ which contains Z. Thus $T_Z \neq (\bigcup W_i)_Z$ and so, by definition, T_Z is not of countable union type. Thus the theorem is true.

Also, consider Example α_2 of [4, p. 244] and Figure 3 of [4, p. 254]. Let C be a nondense and perfect subset on the segment [(1, 0), (1,1)]; for each $c \in C$, let L(c) be the set of all points $(x, y) \in E_2$ which satisfy the equation $y = c + x^{-1} \sin \pi/x$ for $0 < x \leq 1$. Let M be the closure of $\bigcup L(c)$. Let $Z = \{(x, y) \mid x = 0\}$. We see that M is a continuum with the set Z of indecomposability. Let $z \in Z$; then the closure of each L(c) is contained in T_z; hence $T_z = M$. However, $(\bigcup W_i)_z = Z$, and so $T_z \neq (\bigcup W_i)_z$. Thus T_z is not of countable union type; this means that T_z is not the union of a countable number of proper subcontinua of M each of which is not region-containing; this is contrary to the nature of a composant of an indecomposable continuum. It is easy to modify this continuum so that M is compact.

In connection with the example $M = \bigcup I_c$ $(c \in C)$, the following is of interest. Since each $I_c \cup I_e$ is a non-region-containing subcontinuum of M which contains Z, $T_Z = M$. Let $\{R_i\}$ be a countable basis of regions for M; Let W_i' be the maximal subcontinuum of $M - R_i$ which contains Z. Then $\bigcup W_i'$ contains the union of the set of composants for Z of each I_c; it also contains each I_c, where $I_c \cap R_i = \emptyset$. Therefore $\bigcup W_i' = M = T_Z$. The W_i' are region-containing; a nonnull W_i is not (see [5, p. 75, Theorem 107]). Let $M = I_c \cup I_e$ $(c \neq e)$. In the sense of [5, p. 75], the composant of M with respect to Z is M itself; our T_Z is the union of the composant of I_c which contains Z and of the composant of I_e with this property.

LEMMA 6. *Let* M *be a compact continuum with the set* Z *of indecomposability. If* T_p *and* T_q *are both composants of countable union type of* M, *and* $T_p \cap T_q \neq \emptyset$, *then* $T_p = T_q$; *also,* $M - T_p$ *is dense in* M.

Proof. Let $x \in T_p \cap T_q$. By definition of composant, there exist $T_{px} \subset T_p$ and $T_{px} \subset T_q$. Since T_{px} and T_{qx} are each non-region-containing, we see that $M - T_{px}$ and $M - T_{qx}$ are each dense in M. By Theorem 15 of [5, p. 11], the continuum $T_{px} \cup T_{qx}$ must be non-region-containing. Hence $q \in T_p$. If $y \in T_q$, we see, in a similar manner, that $y \in T_q$. Therefore $T_p = T_q$.

Since T_p is of countable union type, $T_p = (\bigcup W_i)_p$. By definition of W_i, W_i does not contain Z, and so W_i is not region-containing. Therefore $M - W_i$ is dense in M. By Theorem 15 of [5, p. 11], $M - T_p$ is dense in M.

THEOREM 7. *Let* M *be a compact continuum with the set* Z *of indecomposability and, for each* $p \in M$, *let* T_p *be of countable union type. Then* M *is the union of an uncountable number of disjoint composants, each of which is dense in* Z; *also, each* $M - T_p$ *is dense in* M.

Proof. This follows at once from Theorem 15 of [5, p. 11], Lemma 6, and Theorem 5.

COROLLARY 7.1. *Let* M *be a compact continuum with the set* Z *of indecomposability, and let* Z *contain an open subset of* M. *Then* M *is the union of uncountably many disjoint composants; each composant* T_p *is dense in* Z, *and* $M - T_p$ *is dense in* M.

Proof. This follows from Theorems 6 and 7.

We say that a continuum M is *irreducible, with respect to a subset* Z, *between two points* a *and* b ($a, b \in M$) if and only if every T_{ab} of M is region-containing and contains Z.

COROLLARY 7.2. *Let* M *be as in Corollary* 7.1. *Then there exist* $a, b, c \in M$ *such that* M *is irreducible with respect to* Z, *between each pair of these.*

Proof. Take a, b, and c each from a different composant of M. Then there does not exist a non-region-containing T_{ab}. Hence each $T_{ab} \supset Z$. Therefore the Corollary is true.

THEOREM 8. *Let* M *be a compact continuum which is the union of continua* M_i ($i = 1, 2, \cdots, n$; $n \geq 2$), *where each* M_i *is a continuum with a set* Z_i *of indecomposability, and* Z_i *contains an open subset of* M_i; *and where also, for* $i \neq j$, $M_i \cap M_j$ *consists of at most a countable number of components, each of which is not region-containing. Then there exist* $p_i \in M_i$ *such that any subcontinuum* W *of* M *which contains* $\bigcup p_i$ *also contains* $\bigcup Z_i$. *The possible* $p_i \in M_i$ *are dense in* M_i.

Proof. By Corollary 7.1, each M_i contains uncountably many composants. Each pair M_i and M_j has the property that $M_i \cap M_j$ consists of a countable number of components, and that each of these components is in a T_p of M_i and also in a T_p' of M_j. The set of all composants that contain points of two M_i is countable. Take p_i in M_i, but in none of this countable set of composants. Then any subcontinuum of M_i which contains both p_i and a point of an M_j ($j \neq i$) must contain Z_i, because of the proof of Corollary 7.2. Thus we see that the theorem is true.

Let Z be a subset of the connected set W. We say that W is *widely connected with respect to* Z if and only if every nondegenerate connected subset V of W is such that $\overline{V} \supset \overline{Z}$. An example is a biconnected set W with dispersion point Z.

THEOREM 9. *If* M *is a compact continuum with the set* Z *of indecomposability, and* Z *contains an open subset of* M, *then, under the continuum hypothesis,* M *contains a set* W *which is widely connected with respect to* Z *and satisfies the condition* $\overline{W} = \overline{M}$.

Proof. We follow the proof of Theorem 1 of [6, pp. 254-256]. Let $\{B_\alpha\}$ be the class of separating boundary continua of M; it is known to be of power $2^{\aleph_0}$; also, by Corollary 7.1, the class $\{T_p\}$ of disjoint composants of M is of this power. Hence we can take p_α in B_α and p_α in some T_p. We see that if H is a domain bounded by B_α and $p \in H$, but H does not contain all of Z, then $T_p \cap B_\alpha \neq \emptyset$; for since B_α bounds at least two domains, we see by Corollary 7.1, that there exist uncountably many disjoint composants T_x such that $T_x \cap B_\alpha \neq \emptyset$. Therefore each p_α can be taken in a different T_p. Let $W = \bigcup p_\alpha$ ($\alpha < \Omega$, where Ω is the first ordinal with power $2^{\aleph_0}$). Then W is connected, because it contains a point of each B_α. It contains but one point of each T_p; hence each nondegenerate connected subset V of W is not contained in any T_p, and therefore $\overline{V}$ must be a region-containing subcontinuum of M. Therefore $\overline{V} \supset \overline{Z}$, and hence W is widely connected with respect to Z. because each region with respect to M contains a $B_\alpha \cap M$, $\overline{W} = \overline{M}$.

THEOREM 9.1. *Let* W *be a widely connected set with respect to* Z, *and let* Z *contain a subset* H *open with respect to* M, *where* M *is as in Theorem* 9. *Then* W - H *is totally disconnected.*

Proof. Suppose that W - H contains the nondegenerate connected subset C. Then $\overline{C} \supset \overline{Z}$ and $Z \supset H$. Thus there exists a region R such that $\overline{C} \supset R \cap M$. Hence $C \cap H \neq \emptyset$, which is a contradiction.

11. CONNECTED TOWERS OF WADA

We give here some typical theorems involving towers of Wada domains. Variations of these are easily stated and proved. One can get local connectedness in the chain construction (C.1), even if the connected Wada domain is constructed with branches in its simple chain unions; however, in what follows, we take the chain construction without branching, so as to give the desired type of shielding to $\bigcap D^h$.

THEOREM 10. *Let* $\{D^g\}$ *be an infinite tower of connected Wada domains, of chain construction* (C.1) *or* (C.2), *or of indecomposable network construction* (C.4), *such that each* $\overline{D}^g = \overline{D}$, *and such that* $\bigcap D^g$ *does not contain a domain. Then, for every integer* $n > 2$ *and for* $n = \aleph_0$, D *is the union of* n *disjoint indecomposable connected subsets, each dense in* D, *one of which is* $\bigcap D^g$ *while the others are of the form* $(\bigcup F^f)_e$, *where* $F^f = F(D^f) - F(D^{f-1})$ *and each* f *is some* g, *and where* $\{D^f\}_e \subset \{D^g\}$.

Proof. In the constructions (C.1), (C.2), and (C.4), it is to be noted that the class $\{F(D^g)\}$ gives shielding densely to $\bigcap D^g$. Thus, by Lemma 2, $\bigcap D^g$ is an indecomposable connected set.

Let $D = D^0$. We see that $F^1 = F(D^1) - F(D^0)$ and $F^2 = F(D^2) - F(D^1)$ are disjoint; thus $\{F^g\}$ is a disjoint class. For $n = 2, 3, \cdots, \aleph_0$, transfinite number theory allows us to arrange the class $\{F^g\}$ into n disjoint subclasses $\{F^f\}_e$ $(e = 2, 3, \cdots, n$ or $e = 1, 2, \cdots)$, each with infinitely many elements. Let $(\bigcup F^f)_e$ be the union of the elements of $\{F^f\}_e$. Since each F^h is a limiting set of F^{h+t}, $(\bigcup F^f)_e$ is connected. Because $n > 2$, and in a tower $D^g \supset D^{g+1}$, each $\{F^f\}_e$ shields $(\bigcup F^f)_c$ $(c \neq e)$ densely in (C.1), (C.2), and (C.4). Hence, by Lemma 2, each $(\bigcup F^f)_e$ is an indecomposable connected subset of D, and therefore the theorem is true.

Let M be a connected set, $a, b \in M$. By K_{ab} we mean a connected subset of M such that $K \supset a \cup b$. In [7, p. 797], we defined a *composant* K_p *of* M *with respect to* p by the equality $K_p = [q \mid$ there exist K_{pq} such that $\overline{K}_{pq} \neq \overline{M}]$.

THEOREM 11. *Let* $\{D^g\}$ *be an infinite tower of connected Wada domains, of any mixture of the chain constructions* (C.1) *and* (C.2); *let each* D^g *be dense in the compact domain* D; *and let* $\bigcap D^g$ *be such that it does not contain a domain. Then the indecomposable connected set* $\bigcap D^g$ *has uncountably many disjoint composants, each of which is dense in* D; *if* C *is a connected subset of one of these composants* K_p, *and* $\overline{K}_p \neq \overline{C}$, *then* $K_p \supset \overline{C}$.

Proof. Let $\{R_j\}$ $(j = 1, 2, \cdots)$ be a region basis of $\bigcap D^g$. Let $p \in \bigcap D^g$. Then p is in each D^g. Consider a given R_j. Two cases arise: Either there exists a k such that, for all $g > k$, there exists a simple chain C^g of links, from the construction of D^g, which has its end regions contained in R_j, and one of whose links contains p; or, for $g > k$, there exists the simple chain C^g from p to R_j. Let K_p^g be the union of the chain regions of C^g. Let R be a chain region used in the construction of D^g, and let R' be one of those used in the construction of D^f $(f > g)$. In a Wada construction, if $R \supset R'$, then $R \supset \overline{R}'$. Thus $K_p^g \supset \overline{K}_p^{g+1}$. Let $K(p, R_j) = \bigcap \overline{K}_p^g$. Then $K(p, R_j)$ is a continuum joining p and $\overline{R}_j$, and it is contained in $\bigcap D^g$. Therefore we have the class $\{K(p, R_j)\}$ $(j = 1, 2, \cdots)$; let $K'_p = \bigcup K(p, R_j)$. Then K'_p is a connected subset of $\bigcap D^g$ which is dense in D, and $p \in K'_p$. Let K_p be the composant of $\bigcap D^g$ with respect to p, which we defined above. Thus $K_p \supset K'_p$.

Let C be a connected subset of $\bigcap D^g$, such that $p \in C$; if $\overline{C} \neq \overline{D}$, then, because $\{F(D^g)\}$ shields $\bigcap D^g$ densely, $K'_p \supset \overline{C}$. Therefore $K'_p = K_p$, and any two composants are either equal or disjoint. Each K'_p is the union of a countable number of subcontinua $\bigcap \overline{K}^g_p$, and the $F(D^g)$ are countable; hence, by Theorem 15 of [5, p. 11], the composants are uncountably many in number. Thus the theorem is true.

COROLLARY 11.1. *In each connected and compact domain* D *of* E_m *or of* I_ω, *there exists a widely connected subset* W *such that* $\overline{W} = \overline{D}$.

Proof. This is the same as for the original widely connected set, as given in [6, pp. 254-255]; Theorem 11 is used to obtain the indecomposable connected set $\bigcap D^g$, which is the sum of uncountably many composants.

We note the following: If in Theorem 10 the tower has even infinitely many D^g of the Wilder network type (C.3), the conclusion remains true, provided that each of the subclasses $\{D^f\}_e$ contains infinitely many D^g of the other types, to give the shielding needed in the proof. In a mixture of types in an infinite tower, or of coating in a finite tower, these shielding types predominate in determining the resulting set.

We note also that the conclusion of Theorem 11 remains true if we have a mixture of other types of construction of the D^g, so long as we have infinitely many of the chain types to give the class $\{K^g_p\}$ of that proof: for we need not use a K^g_p for each D^g of $\{D^g\}$.

If in Theorem 10 each D^g is assumed to be disconnected, then $\bigcap D^g$ contains uncountably many indecomposable connected subsets, each a tower intersection of connected Wada domains. If $\bigcap D^g$ contains a domain H, other modifications are obtained with proofs similar to the above.

The definition of n-indecomposable connected sets, needed in Theorem 12, is given in Section 10. For other theorems concerning these sets, see [8], [10], [1] and their references.

THEOREM 12. *Let* $\{D^g\}$ *be a tower of Wada domains such that each* D^g *is contained and dense in a compact connected domain* D *of* E_m;

$$D^g = {}_1D^g \cup {}_2D^g \cup \cdots \cup {}_nD^g;$$

for each j, $\{{}_jD^g\}$ *is a tower of connected Wada domains; each* ${}_jD^g$ *is contained and dense in* ${}_jD^1$; *and each* D^g *is of* (C.2) *chain type construction. Then*

$$\bigcup F(D^g) = F(D^f) \qquad (g = 1, 2, \cdots, f;\ f > 1),$$

and F(D) *is an* n-*indecomposable continuum. Also,* $\bigcap D^g$ $(g = 1, 2, \cdots)$ *is an* n-*indecomposable connected set; in fact,* D *is a disjoint sum, similar to that in the conclusion of Theorem* 10, *with "*n-*indecomposable" substituted for "indecomposable."*

Proof. Since $F(D^1) \subset F(D^2) \subset \cdots \subset F(D^f)$, we see that $\bigcup F(D^f) = F(D^f)$ $(g = 1, 2, \cdots, f;\ f > 1)$. Moreover, $F(D^f) = F({}_1D^f) \cup F({}_2D^f) \cup \cdots \cup F({}_nD^f)$; also, $F({}_jD^f) = F({}_jD^1) \cup F({}_jD^2) \cup \cdots \cup F({}_jD^f)$ for $j = 1, 2, \cdots, n$. Thus $F(D^f)$ is the essential union of the n sets $F({}_jD^f)$ and, by Lemma 3, each $F({}_jD^f)$ is an indecomposable continuum.

For $F(D^f)$ to be an n-indecomposable continuum, the second part of the definition must be satisfied. Suppose that it is not: then $F(D^f)$ is the essential union of the $n + 1$ connected subsets M_i ($i = 1, 2, \cdots, n + 1$). Thus $E(M_i) \neq \emptyset$, for each i. Because each proper subcontinuum of an indecomposable continuum is a continuum of condensation, it follows that $\overline{M}_i$ cannot be a proper subcontinuum of any one $F({}_jD^f)$. Hence two M_i (call them M' and M'') must be such that for one $F({}_jD^f)$, say it is $F({}_1D^f) = {}_1B^f$, it is true that

$$E(M') \cap E({}_1B^f) \neq \emptyset \neq E(M'') \cap E({}_1B^f).$$

Let $q \in E(M') \cap E({}_1B^f)$. For each j, $F({}_jD^1) \supset F(D^1)$. Thus $q \in E({}_1B^f)$ implies that $q \in F(D^1)$ is false. But ${}_1B^f \subset {}_1\overline{D}^2$, and therefore $q \in {}_1D^2$. Thus, by Theorem 34 of [5, p. 21], there exists an irreducible continuum T' in $\overline{M}'$, joining q to $F({}_1D^2)$. Since $T' \subset \overline{M}' \subset F(D^f)$, T' must be contained in $F({}_1D^f)$. Therefore T' and $E(M'')$ cannot be disjoint, and so $E(M') \cap E(M'') \neq \emptyset$. This is a contradiction. Therefore $F(D^f)$ is the essential sum of n, but not of $n + 1$ connected subsets. Hence $F(D^f)$ is an n-indecomposable continuum, which was to be proved.

Consider now the case for $\bigcap D^g$. Each $D^g = {}_1D^g \cup {}_2D^g \cup \cdots \cup {}_nD^g$. By Theorem 10, for each of the values $j = 1, 2, \cdots, n$, the intersection $\bigcap {}_jD^g$ ($g = 1, 2, \cdots$) is an indecomposable connected set. Thus we see easily, from our (C.2) construction, that $\bigcap D^g$ is the essential union of the n indecomposable connected subsets $\bigcap {}_jD^g$ ($j = 1, 2, \cdots, n$). It remains to prove that it is not the essential union of connected subsets M_i ($i = 1, 2, \cdots, n + 1$). The proof is similar to the case above: By Theorem 108' of [7, p. 800], no M_i can be contained entirely in a set $\bigcap {}_jD^g$. Hence two M_i, call them M' and M'', must be such that for one of the $\bigcap {}_jD^g$, say it is $\bigcap {}_1D^g = W$, it is true that $E(M') \cap E(W) \neq \emptyset \neq E(M'') \cap E(W)$. Let $q \in E(M') \cap E(W)$. Then q is contained in some chain region, for each ${}_1D^g$ ($g = 1, 2, \cdots$). Since M' is connected and $\overline{M}' \neq \overline{W}$, M' cannot be contained in any finite number of the chain regions of ${}_1D^g$, for any fixed g. Since part of $E(M')$ is contained in W, it follows, by the proof of Lemma 2, that M' must be dense in W. But then $E(M') \cap E(M'') \neq \emptyset$, which is a contradiction. Therefore $\bigcap D^g$ is an n-indecomposable connected set, and the theorem is true.

THEOREM 13. *Let* Z *be a closed subset of* D. *Then there exists a tower* $\{D^g\}$ *of connected Wada domains such that each* D^g *is of* (C.4) *construction and satisfies the condition* $\overline{D}^g = \overline{D}$; *each* $F(D^g)$ *is a continuum with the set* $Z \cap F(D^g)$ *of indecomposability; and* $\bigcap D^g$ *is a connected set with the set* Z *of indecomposability.*

The proof uses ideas from the proofs of Theorem 1 and 10, and it is not difficult.

In various branches of topology, the more complex examples obtainable by Wada constructions have attracted little attention, either because of too primitive a state of development or for other reasons. In [11], Wilder considered examples which he showed to be treatable by established methods of a unified topology; his introduction there is of some interest concerning the more complex Wada sets above. In part, methods seem lacking for the study, in desired detail, of the more complicated connected sets of Wada; and still more complex types of these sets may yet be discovered.

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