DENSE SUBSETS IN THE SPACES l_p

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It is well known that the space l_p is defined, for $1 \leq p < \infty$, as the linear space of all sequences $x = \{\xi_k\}$ of scalars for which $\Sigma_{k=1}^{\infty} |\xi_k|^p$ is finite. If we set $\|x\| = (\Sigma_{k=1}^{\infty} |\xi_k|^p)^{1/p}$, we get a Banach space. Every linear continuous functional x^* in l_p is determined in one and only one way by a sequence $x^* = \{\alpha_k\}$, with

$$\sum_{k=1}^{\infty} |\alpha_k|^q < \infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

by means of the relation $x^*(x) = \Sigma_{k=1}^{\infty} \alpha_k \xi_k$.

If S *is a subset of* l_p, *and if the only linear continuous functional which vanishes on* S *is the null functional, then* S *determines a dense subspace of* l_p (see [1, p. 57], [5, p. 9], and [6, p. 61]).

Inspired by M. V. Subba Rao's paper [6], we obtain, by means of Dirichlet series, a number of propositions concerning dense linear subsets in l_p.

PROPOSITION 1. *Let* $x = \{\xi_k\} \in l_p$, $p \geq 1$, $\xi_k \neq 0$ *for every* k; *let* $\{s_n\}$ *be a sequence of complex numbers* ($s_n \to \infty$ as $n \to \infty$) *lying in the region* $\Re s > 0$, $|\arg s| \leq \phi < \pi/2$, *and let* $x_n = \{\xi_k e^{-\lambda_k s_n}\}$ $(n = 1, 2, \cdots)$, *where*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty \qquad (k \to \infty).$$

Then the linear manifold determined by $\{x_n\}$ *is dense in* l_p.

Proof. Let $x^* = \{\alpha_k\}$ be a linear continuous functional in l_p such that

$$x^*(x_n) = \sum_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s_n} = 0 \qquad (n = 1, 2, \cdots). \tag{1}$$

Since

$$\sum_{k=1}^{\infty} |\alpha_k \xi_k| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\alpha_k|^q\right)^{\frac{1}{q}} < \infty,$$

the Dirichlet series $\Sigma_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s}$ is absolutely and uniformly convergent in the closed half-plane $\Re s \geq 0$. Hence it represents an analytic function

$$f(s) = \sum_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s} \qquad (s = \sigma + it) \tag{2}$$

which is certainly holomorphic in the half-plane $\sigma > 0$. Furthermore, from (1) we see that the function $f(s)$ has infinitely many zeros $s_1, s_2, \cdots$ lying in an angle

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$|\arg s| < \pi/2$ and tending to ∞. Together with Theorem 6 of [3, p. 6], this implies that $\alpha_k \xi_k = 0$ $(k = 1, 2, \cdots)$. Since $\xi_k \neq 0$ by hypothesis, we conclude that $\alpha_k = 0$ for every k, and $x^* = 0$. This completes the proof of Proposition 1.

PROPOSITION 2. *Let* $x_n = \{(-1)^n \xi_k \lambda_k^n e^{-\lambda_k}\}$ $(n = 0, 1, 2, \cdots)$, *where* $\xi_k \neq 0$ *for every* k *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty$ $(k \to \infty)$. *If* $x = \{\xi_k\} \in l_p$, $p \geq 1$, *then the linear manifold determined by* $\{x_n\}$ *is dense in* l_p.

Proof. Let $x^* = \{\alpha_k\}$ be a linear continuous functional in l_p such that

$$x^*(x_n) = \sum_{k=1}^{\infty} (-1)^n \alpha_k \xi_k \lambda_k^n e^{-\lambda_k} = 0 \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

Since $\Sigma_{k=1}^{\infty} |\alpha_k \xi_k| < \infty$, the series

$$h(s) = \sum_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s}$$

is absolutely and uniformly convergent in the closed half-plane $\sigma \geq 0$. Therefore the function $h(s)$ is an analytic function holomorphic in the half-plane $\sigma > 0$, and its derivatives there are given by the formula

$$h^{(n)}(s) = \sum_{k=1}^{\infty} (-1)^n \alpha_k \xi_k \lambda_k^n e^{-\lambda_k s} \qquad (\sigma > 0;\ n = 0, 1, 2, \cdots).$$

From (3) we see that the function $h(s)$ and its derivatives vanish at $s = 1$. Hence $h(s) \equiv 0$.

We shall now show that $\alpha_k = 0$ $(k = 1, 2, \cdots)$. By the theorem for the evaluation of the coefficients of a Dirichlet series [4, p. 170], which states that

$$\alpha_k \xi_k e^{-\lambda_k \sigma_1} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{T} h(\sigma_1 + it)\, e^{it\lambda_k}\, dt \qquad (k = 1, 2, \cdots),$$

where $\sigma_1 > 0$ and t_0 are arbitrary, the relation $h(s) \equiv 0$ implies that $\alpha_k \xi_k = 0$ $(k = 1, 2, \cdots)$. But no ξ_k vanishes, by hypothesis. Hence $\alpha_k = 0$ for every k. Consequently, $x^* = 0$, and Proposition 2 follows.

PROPOSITION 3. *Let* $x_n = \{(-1)^n \xi_k \lambda_k^n e^{-\lambda_k}\}$ $(n = 0, 1, 2, \cdots)$, *where* $\xi_k \neq 0$ *for every* k *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty$ $(k \to \infty)$. *If*

$$\limsup_{k \to \infty} \frac{\log k}{\lambda_k} = 0 \qquad \textit{and} \qquad \limsup_{k \to \infty} \frac{\log |\xi_k|}{\lambda_k} = 0,$$

then the linear manifold determined by $\{x_n\}$ *is dense in* l_p, *for each* $p \geq 1$.

Proof. According to Theorem VII of [4, p. 166], the series $\Sigma_{k=1}^{\infty} \xi_k e^{-\lambda_k s}$ converges absolutely in the half-plane $\sigma > 0$. Also, each of the derived series $\Sigma_{k=1}^{\infty} (-1)^n \xi_k \lambda_k^n e^{-\lambda_k s}$ $(n = 1, 2, \cdots)$ converges absolutely in this half-plane. For this reason we have, at $s = 1$, $\Sigma_{k=1}^{\infty} |\xi_k| \lambda_k^n e^{-\lambda_k} < \infty$, and $x_n \in l_p$ for each $p \geq 1$. The remainder of the proof is as in Proposition 2.

PROPOSITION 4. *Let* $x_n = \{(-1)^n \xi_k \lambda_k^n\}$ $(n = 0, 1, 2, \cdots)$, *where* $\xi_k \neq 0$ *for every* k *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty$ $(k \to \infty)$. *If*

$$\limsup_{k \to \infty} \frac{\log k}{\lambda_k} = 0 \quad \textit{and} \quad \limsup_{k \to \infty} \frac{\log |\xi_k|}{\lambda_k} = -\infty,$$

then the linear manifold determined by $\{x_n\}$ *is dense in* l_p, *for each* $p \geq 1$.

Proof. By virtue of Theorem VII of [4, p. 166], the series $\Sigma_{k=1}^{\infty} \xi_k e^{-\lambda_k s}$ converges absolutely in the whole plane. Hence, at $s = 0$, we have $\Sigma_{k=1}^{\infty} |\xi_k| \lambda_k^n < \infty$, and $x_n \in l_p$ for each $p \geq 1$. The remainder of the proof is evident from what has been shown in Proposition 2.

PROPOSITION 5. *Let* $x_n = \{(-1)^n \xi_k \lambda_k^n\}$ $(n = 0, 1, 2, \cdots)$, *where* $\xi_k \neq 0$ *for every* k *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty$ $(k \to \infty)$. *If*

$$\limsup_{k \to \infty} \frac{1}{\lambda_k} \log \sum_{\nu=1}^{\nu=k} |\xi_\nu| e^{r\lambda_\nu} \leq 0 \qquad (-\infty < r < \infty), \tag{4}$$

then the linear manifold determined by $\{x_n\}$ *is dense in* l_p, *for each* $p \geq 1$.

Proof. The condition (4) implies the absolute convergence of the series $\Sigma_{k=1}^{\infty} \xi_k e^{-\lambda_k s}$ in the whole plane [2, pp. 7-8]. The remainder of the proof is as in Proposition 4.

PROPOSITION 6. *Let*

$$x_n = \left\{ (-1)^n \frac{\xi_k \lambda_k^n}{\Gamma(1 + \alpha\lambda_k)} \right\} \qquad (n = 0, 1, 2, \cdots),$$

where $\alpha > 0$, $\xi_k \neq 0$ *for every* k, *and* $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k \to \infty$ $(k \to \infty)$. *If* $\Sigma_{k=1}^{\infty} |\xi_k| < \infty$, *then the linear manifold determined by* $\{x_n\}$ *is dense in* l_p, *for each* $p \geq 1$.

Proof. The series $\Sigma_{k=1}^{\infty} \xi_k e^{-\lambda_k s}$ converges uniformly in the half-plane $\sigma \geq 0$. This implies the uniform convergence of the series

$$\sum_{k=1}^{\infty} \frac{\xi_k e^{-\lambda_k s}}{\Gamma(1 + \alpha\lambda_k)}$$

in the whole plane [2, p. 184]. The remainder of the proof is as in Proposition 4.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

2. V. Bernstein, *Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Gauthier-Villars, Paris, 1933.

3. G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge University Press, 1952.

4. S. Mandelbrojt, *Dirichlet series*, Rice Inst. Pamphlet 31 (1944), 159-272.

5. ———, *General theorems of closure*, Rice Inst. Pamphlet, Special Issue (1951), 1-71.

6. M. V. Subba Rao, *Closure theorems*, Math. Student 26 (1958), 61-70.

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