CLOSED METRIC FOLIATIONS

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1. We adopt here the notation and definitions of [5]: M is a C^∞ n-dimensional manifold with a p-dimensional foliation F and a fibrelike Riemannian ds^2. (In view of the results of [3], it seems wise to reserve the term "bundle-like" for the case where the leaves are totally geodesic.) We shall be concerned solely with the case where all leaves are closed subsets of M. The properties mentioned so far will be indicated by saying that M has a *closed metric foliation*. The quotient space $B = M/F$ is the space formed from M by identifying each leaf to a point, and π: M → B is the identification map. If L is a leaf, H(L) is the holonomy group of L.

The concept of V-*manifold* has been defined by Satake [6]; roughly speaking, a V-manifold is a connected Hausdorff space which is locally homeomorphic to the quotient of Euclidean space by a finite, differentiable transformation group. For precise definitions of V-manifold and V-bundle, we refer to Baily [1]. We also introduce the notion of V-*fibre space*. defined by dropping the structural group from the definition of V-bundle. We shall use $\{U, G, \phi\}$ to denote a local uniformizing structure on an open set in the V-manifold B, λ to denote an injection $\{U, G, \phi\} \to \{U', G', \phi'\}$, and h_U to denote an anti-isomorphism of G into a group of fibre mappings of a fibre space B_U over U onto itself. All these objects are assumed to have the properties postulated in [1].

In a previous paper [5], we described the structure of a metric foliation in the neighborhood of a leaf. The description is incorrect for a leaf which is not a closed set [2], but is valid in the case in which we are now interested. (The difficulty lies in the fact that the construction required for the theorem may not be possible in M, but must be carried out in an auxiliary space. Corollary 1 in [5] remains correct in the general case.) Our present aim is to discuss the structure globally.

THEOREM. *A closed metric foliation* F *of a complete Riemannian manifold* M *is a* V-*fibre space over a* V-*manifold* B *as a base space, where* $B = M/F$ *and* π: M → B *is the identification map.*

2. We need two lemmas.

LEMMA 1. *For a closed metric foliation, the holonomy group of each leaf is finite.*

Proof. H(L) may be considered as a group of isometries of the sphere of unit vectors orthogonal to the leaf L at some arbitrary point of L. By the exponential map, this sphere may be embedded in a small (n - p)-plane formed by orthogonal geodesics radiating from this point. The orbit of a point P under H(L) is just the intersection of the leaf through P with this embedded sphere, hence is a closed subset of the sphere. We first show that this orbit is a finite set. Suppose not; then it has a point of accumulation, which belongs to the orbit since it is a closed set. But

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then every point of the orbit is a point of accumulation, that is, the orbit is a perfect set, hence is uncountable, a contradiction. Hence each orbit of H(L) is finite. Let G be the closure of H(L) considered as a subgroup of the orthogonal group. G has a finite number of components. The orbit of any point under the component of the identity G_0 is a connected set. On the other hand, since each orbit is closed, the orbit under G is the same as the orbit under H(L), so is finite. It follows that the orbit of any point under G_0 consists of one point, that is, each element of G_0 induces the identity on the normal sphere. Hence, G_0 is the identity, and G = H(L) is finite.

LEMMA 2. *The quotient space* B *of a closed metric foliation on a complete manifold is a metric space, if we define the distance between two points of* B *to be the minimum distance between them considered as leaves in* M.

This lemma is proved by Hermann [4].

3. We proceed to the proof of the theorem.

B is a connected Hausdorff space, since it is metric and is the continuous image of M under π. Given any point $b \in B$, let $L = \pi^{-1}(b)$. Let W be a flat coordinate neighborhood [5] about some point of L and contained in an ε-neighborhood of L for ε sufficiently small [2, 5]. If we consider the (n - p)-ball swept out in W by geodesics normal to L at some fixed point and of length less than ε_1, then H(L) operates on this ball in such a manner that $\{W, H(L), \pi\}$ is a local uniformizing structure for the neighborhood $\pi(W)$ in B. The natural injection maps of two such structures are differentiable. Since H(L) is an isometry on the normal vectors at a point of L, the normal component of the metric of M defines a Riemannian structure on B. (Notice that this is not the metric induced on W by its embedding into M). We conclude that B is a Riemannian V-manifold, as required.

Let M^0 be the union of all leaves L such that H(L) = 1, and let $B^0 = \pi(M^0)$. Then M^0 is not empty; indeed, the finiteness of all holonomy groups implies that M^0 is dense in M. All the leaves in M^0 are homeomorphic [5, Corollary 1]. We denote a typical one by F, since it will be the fiber of the V-fiber space in question. H(L) operates on F as a group of covering transformations [5], and on W as described above. Hence, we may define an operation of H(L) on $W \times F$ as a differentiable transformation group, according to the rule

$$h_W(g)(w, f) = (g^{-1}w, g^{-1}f),$$

where $g \in H(L)$, $w \in W$, and $f \in F$. Now W was originally constructed as a subset of M, and F may be embedded in $X = \pi^{-1}(\pi(W))$ as a leaf. If we fix a base point in the intersection of these two subsets (which is nonempty and finite), the action of the holonomy group gives rise to a differentiable mapping of $W \times F$ onto X which is of maximal rank everywhere, hence defines a diffeomorphism of $W \times F/H(L)$ onto X. The necessary injection mappings clearly exist, so that M is indeed the total space of a V-fibre space.

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