ON THE HERMITEAN PRODUCT OF ORDERED POINT SETS ON THE UNIT CIRCLE

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1. INTRODUCTION

The following proposition was suggested to us by V. L. Klee, who described it as a conjecture of C. J. Titus and J. L. Ullman.

(1.1) *If* $0 \le \phi_j < 2\pi$ *for* $j = 1, 2, \cdots, n$,

$$0 \le \psi_j < 2\pi \quad \textit{for } j = 1, 2, \cdots, n,$$

$$\phi_j < \phi_{j+1},\ \psi_j < \psi_{j+1} \quad \textit{for } j = 1, \cdots, n-1, \textit{ and}$$

$$\sum_{j=1}^{n} e^{i\phi_j} = \sum_{j=1}^{n} e^{i\psi_j} = 0,$$

then

$$\sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} \neq 0.$$

In Section 2, we prove the conjecture, reducing the problem to the case where $\phi_j \geq \psi_j$ for $j = 1, 2, \cdots, n$. That this can be done was suggested to us by N. H. Kuiper, who proved it independently. In Section 3, we consider the continuous analogue of the problem. In Section 4, we see that if $n \equiv 0 \pmod 4$, the sum considered in Section 1 has 0 as g.l.b., and that the g.l.b. of the sum is positive if $n \not\equiv 0 \pmod 4$. We compute a lower bound for the sum, but do not prove that the value found really is the g.l.b. of $|\Sigma e^{i(\phi_j - \psi_j)}|$.

2. PROOF OF THE CONJECTURE

We first prove the following theorem.

(2.1) *If* $0 = \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$, $0 = \psi_1 < \psi_2 < \cdots < \psi_n < 2\pi$,

$$\phi_j \geq \psi_j \text{ for } j = 1, 2, \cdots, n, \textit{ and}$$

$$\sum_{j=1}^{n} e^{i\phi_j} = \sum_{j=1}^{n} e^{i\psi_j} = 0,$$

then

$$\sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} \neq 0.$$

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Proof.

$$\Im \sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} = \Im\left(\sum_{j=1}^{n} e^{i(\phi_j-\psi_j)} - \sum_{j=1}^{n} e^{i\phi_j} + \sum_{j=1}^{n} e^{i\psi_j} \right)$$

$$= \sum_{j=1}^{n} \{ \sin(\phi_j - \psi_j) - \sin\phi_j + \sin\psi_j \}$$

$$= 4 \sum_{j=1}^{n} \sin\frac{\phi_j}{2} \sin\frac{\psi_j}{2} \sin\frac{\phi_j - \psi_j}{2}.$$

In this sum, the first term is 0 and the following terms are nonnegative. Equality occurs only if $\phi_j = \psi_j$. We have proved that

$$\sum_{j=1}^{n} e^{i(\phi_j-\psi_j)} \textit{ is real if and only if } \phi_j = \psi_j \textit{ for } j = 1, 2, \cdots, n,$$

$$\textit{in which case } \sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} = n.$$

We now prove the conjecture (1.1) stated in the Introduction. We have

$$0 < \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi, \qquad 0 < \psi_1 < \psi_2 < \cdots < \psi_n < 2\pi.$$

Suppose $\min_j (\phi_j - \psi_j) = -\beta$. Then for some index k we have

$$\phi_k - \psi_k = -\beta, \qquad \psi_k = \gamma.$$

Hence

$$0 = \phi_k + \beta - \gamma < \phi_{k+1} + \beta - \gamma < \cdots < \phi_n + \beta - \gamma < \phi_1 + 2\pi + \beta - \gamma < \cdots$$
$$< \phi_{k-1} + 2\pi + \beta - \gamma < 2\pi.$$

and

$$0 = \psi_k - \gamma < \psi_{k+1} - \gamma < \cdots < \psi_n - \gamma < \psi_1 + 2\pi - \gamma < \cdots < \psi_{k-1} + 2\pi - \gamma < 2\pi.$$

If we call the terms in these two series of inequalities α_j and β_j, respectively, we have

$$0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n < 2\pi, \qquad 0 = \beta_1 < \beta_2 < \cdots < \beta_n < 2\pi,$$

$$\alpha_j - \beta_j = \phi_{\ell_j} - \psi_{\ell_j} + \beta \geq 0,$$

$$\sum_{j=1}^{n} e^{i\alpha_j} = e^{i(\beta-\gamma)} \sum_{\ell=1}^{n} e^{i\phi_\ell} = 0, \qquad \sum_{j=1}^{n} e^{i\beta_j} = e^{-i\gamma} \sum_{\ell=1}^{n} e^{i\psi_\ell} = 0,$$

$$\sum_{j=1}^{n} e^{i(\alpha_j - \beta_j)} = e^{i\beta} \sum_{\ell=1}^{n} e^{i(\phi_\ell - \psi_\ell)}.$$

By (2.1), we have $\sum_{j=1}^{n} e^{i(\alpha_j - \beta_j)} \neq 0$ and hence $\sum_{\ell=1}^{n} e^{i(\phi_\ell - \psi_\ell)} \neq 0$, which proves the conjecture.

3. GENERALIZATION TO SEMICONTINUOUS FUNCTIONS ON THE UNIT CIRCLE

We now give a generalization of Theorem (1.1).

(3.1) *If* ϕ *and* ψ *are two monotonic right-continuous functions defined for* $0 \leq t \leq 1$ *for which* $0 \leq \phi(t) \leq 2\pi$, $0 \leq \psi(t) \leq 2\pi$, *and* $\int_0^1 e^{i\phi(t)}\,dt = \int_0^1 e^{i\psi(t)}\,dt = 0$, *then* $\int_0^1 e^{i\{\phi(t) - \psi(t)\}}\,dt = 0$ *if and only if* $\phi(t)$, $\psi(t)$ *is one of the following pairs of functions:*

$$\phi_1(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{1}{4}, \\ \pi & \text{for } \frac{1}{4} \leq t < \frac{3}{4}, \\ 2\pi & \text{for } \frac{3}{4} \leq t \leq 1, \end{cases} \qquad \psi_1(t) = \begin{cases} \alpha & \text{for } 0 \leq t < \frac{1}{2} \ (0 \leq \alpha \leq \pi), \\ \pi + \alpha & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\phi_2(t) = \begin{cases} 0 & \text{for } 0 \leq t < T \leq \frac{1}{4}, \\ \pi & \text{for } T \leq t < T + \frac{1}{2}, \\ 2\pi & \text{for } T + \frac{1}{2} \leq t \leq 1, \end{cases} \qquad \psi_2(t) = \begin{cases} 0 & \text{for } 0 \leq t < T + \frac{1}{4}, \\ \pi & \text{for } T + \frac{1}{4} \leq t < T + \frac{3}{4}, \\ 2\pi & \text{for } T + \frac{3}{4} \leq t \leq 1 \end{cases}$$

(of course, it does not matter which of the two functions we call ϕ and which ψ).

Proof. Suppose $\inf\{\phi(t) - \psi(t)\} = -\beta$. Then for some t_0 we have

$$\phi(t_0) - \psi(t_0) = -\beta \qquad \text{or} \qquad \phi(t_0 - 0) - \psi(t_0 - 0) = -\beta.$$

In the first case take $\gamma = \psi(t_0)$, in the second case, $\gamma = \psi(t_0 - 0)$. Consider the functions $\alpha(t)$ and $\beta(t)$ defined in the following way:

$$\alpha(t) = \begin{cases} \phi(t_0 + t) + \beta - \gamma & \text{for } 0 \le t < 1 - t_0, \\ \phi(t + t_0 - 1) + 2\pi + \beta - \gamma & \text{for } 1 - t_0 \le t < 1, \end{cases}$$

$$\alpha(1) = \phi(t_0 - 0) + 2\pi + \beta - \gamma,$$

$$\beta(t) = \begin{cases} \psi(t_0 + t) - \gamma & \text{for } 0 \le t < 1 - t_0, \\ \psi(t + t_0 - 1) + 2\pi - \gamma & \text{for } 1 - t_0 \le t < 1, \end{cases}$$

$$\beta(1) = \psi(t_0 - 0) + 2\pi - \gamma.$$

The functions $\alpha(t)$ and $\beta(t)$ satisfy the same conditions as $\phi(t)$ and $\psi(t)$ in (3.1), and also $\alpha(t) \ge \beta(t)$ for $0 \le t \le 1$. Furthermore $\alpha(0) = \beta(0) = 0$ or $\alpha(1) = \beta(1) = 2\pi$.

Suppose $\int_0^1 e^{i\{\phi(t) - \psi(t)\}}\, dt = 0$. Then we have

$$\int_0^1 e^{i\{\alpha(t) - \beta(t)\}}\, dt = e^{i\beta} \int_0^1 e^{i\{\phi(t) - \psi(t)\}}\, dt = 0.$$

In the same way as in Section 2, we find

(a) $$\int_0^1 \sin \frac{\alpha(t)}{2} \sin \frac{\beta(t)}{2} \sin \frac{\alpha(t) - \beta(t)}{2}\, dt = 0.$$

A consequence of (a) is that there must exist numbers $\tau_1, \tau_2, \tau_3, \tau_4$ with $0 \le \tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le 1$ such that

(b) $$\begin{cases} \alpha(t) = \beta(t) = 0 & \text{for } 0 \le t < \tau_1, \\ \alpha(t) > 0,\ \beta(t) = 0 & \text{for } \tau_1 \le t < \tau_2, \\ \alpha(t) = \beta(t) & \text{for } \tau_2 \le t < \tau_3, \\ \alpha(t) = 2\pi,\ \beta(t) < 2\pi & \text{for } \tau_3 \le t < \tau_4, \\ \alpha(t) = \beta(t) = 2\pi & \text{for } \tau_4 \le t < 1. \end{cases}$$

Now, if $\tau_2 = \tau_3$, we have a contradiction from this relation,

$$1 = \int_0^1 [1 + \cos\{\alpha(t) - \beta(t)\} - \cos \alpha(t) - \cos \beta(t)]\, dt$$

$$= 4 \int_0^1 \sin \frac{\alpha(t)}{2} \sin \frac{\beta(t)}{2} \cos \frac{\alpha(t) - \beta(t)}{2}\, dt = 0.$$

Hence $\tau_2 < \tau_3$.

Since $\int_0^1 \sin\alpha(t)\,dt = \int_0^1 \sin\beta(t)\,dt = 0$, we see that

$$\alpha(\tau_2) = \beta(\tau_2) \le \pi \le \alpha(\tau_3 - 0) = \beta(\tau_3 - 0).$$

Using (a) and (b), we find that

$$\int_{\tau_1}^{\tau_2} \sin\alpha(t)\,dt = \int_{\tau_3}^{\tau_4} \sin\beta(t)\,dt,$$

and this is possible only if $\alpha(t) = \pi$ for $\tau_1 \le t < \tau_3$ and $\beta(t) = \pi$ for $\tau_2 \le t < \tau_4$. Since

$$\int_0^1 \cos\alpha(t)\,dt = \int_0^1 \cos\beta(t)\,dt = \int_0^1 \cos\{\alpha(t) - \beta(t)\}\,dt = 0,$$

we must have $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \tau_4 - \tau_3 = \frac{1}{4}$. Returning to $\phi(t)$ and $\psi(t)$, we find the result stated in Theorem (3.1).

4. THE g.l.b. OF $\left|\sum_{j=1}^{n} e^{i(\phi_j-\psi_j)}\right|$

Now that we know that $\Sigma_{j=1}^{n} e^{i(\phi_j-\psi_j)} \neq 0$ for the ordered sequences $\{\phi_j\}$ and $\{\psi_j\}$, we consider it of interest to find the greatest lower bound m_n of the absolute value of this sum. The sum is a function of 2n variables which are subject to the constraints stated in (1.1). Treating the problem with Lagrange multipliers does not lead to results. The minimal value we seek is to be found on the boundary of the region we consider. To every set $\{\phi_j\}$ $(j = 1, \cdots, n)$ and $\{\psi_j\}$ $(j = 1, \cdots, n)$ correspond two step functions ϕ and ψ belonging to the class of functions considered in Section 3. We now take these ϕ_j and ψ_j in such a way that their step functions correspond as closely as possible to the two functions ϕ and ψ of Section 3, for which $\int_0^1 e^{i(\phi-\psi)}\,dt = 0$.

A consequence of Section 3 is that m_n can only be 0 if $n \equiv 0 \pmod 4$. If $n = 4k$, take

$$\phi_1 = \phi_2 = \cdots = \phi_k = 0; \quad \phi_{k+1} = \cdots = \phi_{3k} = \pi; \quad \phi_{3k+1} = \cdots = \phi_{4k} = 2\pi,$$

$$\psi_1 = \psi_2 = \cdots = \psi_{2k} = 0; \quad \psi_{2k+1} = \cdots = \psi_{4k} = \pi.$$

Then $m_n = 0$.

For other values of n we find the following estimates for m_n:

<u>$n = 4k + 2$</u>: Let

$$\phi_1 = \cdots = \phi_{k+1} = 0; \quad \phi_{k+2} = \cdots = \phi_{3k+2} = \pi; \quad \phi_{3k+3} = \cdots = \phi_{4k+2} = 2\pi,$$

$$\psi_1 = \cdots = \psi_{2k+1} = 0; \quad \psi_{2k+2} = \cdots = \psi_{4k+2} = \pi.$$

Then $\Sigma = 2$.

n = 4k + 1: Let

$$\phi_1 = \cdots = \phi_k = 0; \quad \phi_{k+1} = \cdots = \phi_{3k} = \pi - \arccos\left(1 - \frac{1}{8k^2}\right);$$

$$\phi_{3k+1} = \pi + \arccos\frac{1}{4k}; \quad \phi_{3k+2} = \cdots = \phi_{4k+1} = 2\pi;$$

$$\psi_1 = \cdots = \psi_{2k} = 0; \quad \psi_{2k+1} = \cdots = \psi_{4k} = \pi - \arccos\left(1 - \frac{1}{8k^2}\right);$$

$$\psi_{4k+1} = \pi + \arccos\frac{1}{4k}.$$

Then $|\Sigma| = \frac{4k+1}{4k}\left(5 - \frac{1}{k}\right)^{1/2} = \frac{n}{n-1}\left(5 - \frac{4}{n-1}\right)^{1/2}$.

n = 4k + 3: Let

$$\phi_1 = \cdots = \phi_{k+1} = 0; \quad \phi_{k+2} = \cdots = \phi_{3k+2} = \pi - \arccos\left(1 - \frac{1}{8\left(k+\frac{1}{2}\right)^2}\right);$$

$$\phi_{3k+3} = \pi + \arccos\frac{1}{4k+2}; \quad \phi_{3k+4} = \cdots = \phi_{4k+3} = 2\pi;$$

$$\psi_1 = \cdots = \psi_{2k+1} = 0; \quad \psi_{2k+2} = \cdots = \psi_{4k+2} = \pi - \arccos\left(1 - \frac{1}{8\left(k+\frac{1}{2}\right)^2}\right);$$

$$\psi_{4k+3} = \pi + \arccos\frac{1}{4k+2}.$$

Then $|\Sigma| = \frac{4k+3}{4k+2}\left(5 - \frac{1}{(2k+1)^2}\right)^{1/2} = \frac{n}{n-1}\left(5 - \frac{4}{(n-1)^2}\right)^{1/2}$.

We did not succeed in proving that these values are the minimal values we seek, although this seems very probable and may well be easy to prove. So we have

(4.1) *The greatest lower bound of* $\left|\Sigma_{j=1}^n e^{i(\phi_j - \psi_j)}\right|$, *where* ϕ_j *and* ψ_j *satisfy the conditions of* (1.1), *is* 0 *if* $n \equiv 0 \pmod 4$.

(4.2) CONJECTURE. *If* $n \equiv 2 \pmod 4$, *then* $\left|\Sigma_{j=1}^n e^{i(\phi_j - \psi_j)}\right| > 2$, *and if* $n \equiv 1 \pmod 2$, *then* $\left|\Sigma_{j=1}^n e^{i(\phi_j - \psi_j)}\right| > \sqrt{5}$, *if* ϕ_j *and* ψ_j *satisfy the conditions of* (1.1).

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