ON GROUP ALGEBRAS OF p-GROUPS

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1. INTRODUCTION

Let G be group, and K a field of characteristic $p \neq 0$. The group algebra Γ_G of G over K consists of all formal sums $\Sigma\, \alpha(g)\, g$, where $g \in G$, $\alpha(g) \in K$, and $\alpha(g) = 0$ for all but finitely many g. The operations + and $\cdot$ are defined in the natural way. Denote by Δ_G the set of all those sums $\Sigma\, \alpha(g)\, g$ for which $\Sigma\, \alpha(g) = 0$; Δ_G is an ideal of Γ_G, generally called the fundamental ideal. Jennings [2] and Lombardo-Radice [3] have both shown that Δ_G is nilpotent if G is a finite p-group. In this paper, we intend to show that the converse is also true. These results will then be applied to the case where G is a locally finite p-group.

The situation where the fundamental sequence $\Delta_G \supseteq \Delta_G^2 \supseteq \Delta_G^3 \supseteq \cdots$ terminates in a finite number of steps at an ideal different from zero appears to be more difficult to analyze. Here, we shall only consider the case where G has exponent p and K is Z_p, the ring of integers modulo p.

2. NILPOTENCE OF THE FUNDAMENTAL IDEAL

LEMMA 2.1. *The elements* $g - 1$ *for all* $g \neq 1$ *in* G *are a basis for* Δ_G. *If* $(h_i)_{i \in I}$ *is a set of generators for* G, *then the subalgebra of* Γ_G *generated by the elements* $h_i^{\pm 1} - 1$ *is exactly* Δ_G. *In fact, the left ideal of* Γ_G *generated by the elements* $h_i - 1$ *is* Δ_G.

Proof. If $\Sigma\, \alpha(g)\, g \in \Delta_G$, then $\Sigma\, \alpha(g) = 0$ and therefore

$$\sum \alpha(g)\, g = \sum \alpha(g)\, g - \sum \alpha(g) = \sum \alpha(g)(g - 1).$$

Hence, the elements $g - 1$ span Δ_G. It is clear that they are linearly independent.

Let $(h_i)_{i \in I}$ be a set of generators for G, and let J be the subalgebra generated by all $h_i^{\pm 1} - 1$. Clearly, $J \subseteq \Delta_G$. If $g \in G$, then

$$g - 1 = h_{i(1)}^{\varepsilon(1)} \cdots h_{i(k)}^{\varepsilon(k)} - 1,$$

where $\varepsilon(j) = \pm 1$. Applying the identity

$$XY - 1 = (X - 1) + (Y - 1) + (X - 1)(Y - 1)$$

to the right-hand side of this equation sufficiently often, we obtain a representation of $g - 1$ as a linear combination of products of the $h_i^{\varepsilon} - 1$. Hence $g - 1$ is in J, and hence $\Delta_G \subseteq J$. Therefore, $J = \Delta_G$.

Let Λ be the left ideal generated by the elements $h_i - 1$. Then

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Received March 24, 1960.

This paper is based in part on the author's doctoral dissertation, University of Michigan, 1958.

$$-h_i^{-1}(h_i - 1) = h_i^{-1} - 1$$

belongs to Λ. Hence, by the above, $\Lambda = \Delta_G$.

THEOREM 2.2. *If* Δ_G *is nilpotent, then* G *is a finite* p*-group.*

Proof. Suppose $\Delta_G^n \neq 0$ and $\Delta_G^{n+1} = 0$. Let $\Sigma\,\alpha(g)\,g$ be a nonzero element of Δ_G^n. Then, for any $h \in G$,

$$\left(\sum \alpha(g)\,g\right)(h - 1) = 0 \qquad \text{or} \qquad \sum \alpha(g)\,g = \sum \alpha(g)\,gh\,.$$

Changing indices on the right gives

$$\sum \alpha(g)\,g = \sum \alpha(gh^{-1})g\,.$$

Consequently, $\alpha(g) = \alpha(gh^{-1})$ for every $g \in G$. Taking $g = h$, we obtain $\alpha(h) = \alpha(1)$ for every $h \in G$. Thus, all the coefficients $\alpha(g)$ are the same, and hence nonzero. But only a finite number of $\alpha(g)$ are nonzero. Therefore G must be finite.

Let $p^k > n$; then $(g - 1)^{p^k} = g^{p^k} - 1 = 0$, since K has characteristic p. Hence, $g^{p^k} = 1$ and therefore G is a finite p-group.

If the field K has characteristic 0, then the analogous situation cannot occur, that is, Δ_G is never nilpotent. For if it were, then G would have to be finite as above. Suppose $g \neq 1$ and $g^m = 1$; then

$$1 = g^m = [(g - 1) + 1]^m = \sum_{j=0}^{m} \binom{m}{j} (g - 1)^j\,,$$

in other words,

$$g - 1 = -\frac{1}{m} \sum_{j=2}^{m} \binom{m}{j} (g - 1)^j\,.$$

Since $g - 1 \in \Delta_G$, $g - 1$ also belongs to Δ_G^2 and thus to $\Delta_G^4, \Delta_G^8, \cdots$. Therefore $g - 1 \in \bigcap \Delta_G^n$. But this contradicts the assumption that Δ_G is nilpotent.

THEOREM 2.3. G *is a locally finite* p*-group if and only if* Δ_G *is locally nilpotent.*

Proof. Let G be a locally finite p-group, and Λ a subring of Δ_G generated by finitely many elements $A_i = \Sigma\,\alpha_i(g)(h - 1)$ $(i = 1, 2, \cdots, k)$. Let S be the set of all g such that $\alpha_i(g) \neq 0$ for at least one $i = 1, \cdots, k$. Then S is finite, and hence the subgroup H generated by S is a finite p-group. Δ_H is the subring of Δ_G generated by all $g^{\pm 1} - 1$ for $g \in S$, by Lemma 2.1. Therefore $\Lambda \subseteq \Delta_H$. By the converse of Theorem 2.2, Δ_H is nilpotent. Hence, Λ is also nilpotent.

Conversely, assume that Δ_G is locally nilpotent. Let H be a finitely generated subgroup of G, generated by $h_1, h_2, \cdots, h_n$. Then Δ_H is finitely generated by $h_i^{\pm 1} - 1$. Consequently, Δ_H is nilpotent, and therefore H is a finite p-group. Therefore G is a locally finite p-group.

3. THE FUNDAMENTAL SEQUENCE FOR GROUPS OF EXPONENT p

In this section, we shall assume that G is a group of exponent p (in other words, that every element has order p) and that the field of coefficients is Z_p.

THEOREM 3.1. *Let* G *be a group of exponent* p, *and let* Δ_G *be the fundamental ideal of the group algebra of* G *over* Z_p. *If* $\Delta_G^n = \Delta_G^{n+1}$, *then*

$$G_{2n+2} = G_{2n+3},$$

where G_i *is the* i*th subgroup in the lower central series of* G.

The proof of this theorem will be based on a modification of a construction due to Grün [1]. We set

$$A_m = G_{m+1}/(G_m, G_{m+1}),$$

where by (H, K) we mean the subgroup of G generated by all commutators $(h, k) = h^{-1}k^{-1}hk$ $(h \in H,\ k \in K)$. Since $(G_{m+1}, G_{m+1}) \subseteq (G_m, G_{m+1})$, A_m is an abelian group of exponent p. We shall therefore regard A_m as an additive Z_p-module; that is, if we set $h_0 = h(G_m, G_{m+1})$, then

i) $1_0 = 0$,

ii) $(gh)_0 = g_0 + h_0$,

iii) $(h^{-1})_0 = -h_0$,

iv) $(h^n)_0 = nh_0$ for $n \in Z_p$.

We allow the elements of G to operate on A_m in the following manner: $h_0 \cdot g = (g^{-1}hg)_0$. It is not difficult to show that the operation is well defined and that G acts as a group of operators on A_m. Since A_m is already a Z_p-module, an operation of Γ_G on A_m may be defined:

$$h_0 \cdot \left(\sum \alpha(g)g\right) = \sum \alpha(g)(h_0 \cdot g).$$

LEMMA 3.2. *If* $h_0 \in A_m$ *and* $g_1, g_2, \cdots, g_k \in G$, *then*

$$(h, g_1, \cdots, g_k)_0 = h_0 \cdot (g_1 - 1) \cdots (g_k - 1).$$

By $(h, g_1, \cdots, g_k)$ we mean the left normed commutator $((\cdots((h, g_1), g_2), \cdots), g_k)$.

Proof. If k = 1, then

$$(h, g_1)_0 = (h^{-1}g_1^{-1}hg_1)_0 = (h^{-1})_0 + (g_1^{-1}hg_1)_0$$

$$= -h_0 + h_0 \cdot g_1 = h_0 \cdot (g_1 - 1).$$

The induction step is trivial.

LEMMA 3.3. $A_m \cdot \Delta_G^n = (G_{m+n+1})_0$; *in particular,* $A_m \cdot \Delta_G^m = 0$.

Proof. If we recall that $h_0 \in A_m$ implies that $h \in G_{m+1}$, this follows immediately from the preceding lemma.

We can now prove Theorem 3.1. If $\Delta_G^n = \Delta_G^{n+1}$, then $A_{n+1} \cdot \Delta_G^n = A_{n+1} \cdot \Delta_G^{n+1} = 0$. Hence, $(G_{2n+1})_0 = 0$, that is, $G_{2n+2} \subseteq (G_{n+1}, G_{n+2}) \subseteq G_{2n+3}$. Therefore $G_{2n+2} = G_{2n+3}$.

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