A NOTE ON THE LAGUERRE POLYNOMIALS

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1. Burchnall [2] employed the operational formula

$$(1) \qquad (D - 2x)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r ,$$

where $D = d/dx$ and

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} ,$$

to prove the identity [3, p. 31]

$$(2) \qquad H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-2)^r \binom{m}{r} \binom{n}{r} r!\, H_{m-r}(x) H_{n-r}(x) .$$

Put

$$(3) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x}) ,$$

the Laguerre polynomial of degree n. Corresponding to (1) we shall show that

$$(4) \qquad \prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{r=0}^{n} \frac{1}{r!} x^r L_{n-r}^{(\alpha+r)}(x) D^r .$$

Note that the linear operators on the left of (4) commute.

To prove (4) we show first that

$$(5) \qquad \Omega_n y = x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x} y) ,$$

where y is an arbitrary (differentiable) function of x and

$$\Omega_n = \prod_{j=1}^{n} (xD - x + \alpha + j), \qquad \Omega_0 = 1 .$$

Clearly (5) holds for $n = 0$. Now

$$\begin{aligned} x^{-\alpha} e^x D^{n+1} (x^{\alpha+n+1} e^{-x} y) &= x^{-\alpha} e^x D^{n+1} (x \cdot x^{\alpha+n} e^{-x} y) \\ &= x^{-\alpha} e^x \{ x D^{n+1} (x^{\alpha+n} e^{-x} y) + (n+1) D^n (x^{\alpha+n} e^{-x} y) \} \\ &= x^{-\alpha} e^x \{ x D (x^{\alpha} e^{-x} \Omega_n y) + (n+1) x^{-\alpha} e^x \Omega_n y \} \end{aligned}$$

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$$= x^{-\alpha} e^{x}\{x[\alpha x^{\alpha-1} e^{-x} \Omega_n y - x^{\alpha} e^{-x}\Omega_n y + x^{\alpha} e^{-x} D\Omega_n y] + (n+1)x^{-\alpha} e^{x} \Omega_n y\}$$

$$= xD\Omega_n y + (\alpha - x + n + 1)\Omega_n y$$

$$= (xD - x + \alpha + n + 1)\Omega_n y$$

$$= \Omega_{n+1} y.$$

This evidently proves (5).

Next, since

$$D^n(x^{\alpha+n} e^{-x} y) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(x^{\alpha+n} e^{-x}) \cdot D^r y$$

$$= \sum_{r=0}^{n} \binom{n}{r} (n-r)!\, x^{\alpha+r} e^{-x} L_{n-r}^{(\alpha+r)}(x)\, D^r y$$

$$= n!\, x^{\alpha} e^{-x} \sum_{r=0}^{n} \frac{1}{r!} x^r L_{n-r}^{(\alpha+r)}(x)\, D^r y,$$

(5) becomes

$$\Omega_n y = n! \sum_{r=0}^{n} \frac{1}{r!} x^r L_{n-r}^{(\alpha+r)}(x)\, D^r y.$$

This completes the proof of (4).

As a special case of (4) we note that

$$n!\, L_n^{(\alpha)}(x) = \prod_{j=1}^{n} (xD - x + \alpha + j) \cdot 1, \tag{6}$$

so that

$$n L_n^{(\alpha)}(x) = (xD - x + \alpha + n) L_{n-1}^{(\alpha)}(x),$$

which is also implied by the familiar recurrences satisfied by $L_n^{(\alpha)}(x)$.

2. We now consider

$$(m+n)!\, L_{m+n}^{(\alpha)}(x) = \prod_{j=1}^{m} (xD - x + \alpha + n + j) \cdot \prod_{k=1}^{n} (xD - x + \alpha + k) \cdot 1$$

$$= n! \prod_{j=1}^{m} (xD - x + \alpha + n + j) \cdot L_n^{(\alpha)}(x)$$

$$= m!\,n! \sum_{r=0}^{m} \frac{1}{r!} x^r L_{m-r}^{(\alpha+n+r)}(x)\, D^r L_n^{(\alpha)}(x).$$

But since

$$D^r L_n^{(\alpha)}(x) = (-1)^r L_{n-r}^{(\alpha+r)}(x),$$

it follows at once that

$$\binom{m+n}{m} L_{m+n}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{(-1)^r}{r!} x^r L_{m-r}^{(\alpha+n+r)}(x)\, L_{n-r}^{(\alpha+r)}(x). \tag{7}$$

From (7) we get

$$\sum_{m=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x)\, t^m = \sum_{r=0}^{n} (-1)^r \frac{(xt)^r}{r!} L_{n-r}^{(\alpha+r)}(x) \sum_{m=0}^{\infty} L_m^{(\alpha+n+r)}(x)\, t^m$$

$$= \sum_{r-0}^{n} (-1)^r \frac{(xt)r}{r!} L_{n-r}^{(\alpha+r)}(x) \cdot (1-t)^{-\alpha-n-r-1} \exp\frac{-xt}{1-t}.$$

Since

$$\sum_{r=0}^{n} (-1)^r \frac{(xt)^r (1-t)^{-r}}{r!} L_{n-r}^{(\alpha+r)} = L_n^{(\alpha)}\left(x + \frac{xt}{1-t}\right),$$

we get

$$\sum_{m=0}^{\infty} \binom{m+n}{m} L_{m+n}^{(\alpha)}(x)\, t^m = (1-t)^{-\alpha-n-1} L_n^{(\alpha)}\left(\frac{x}{1-t}\right) \exp\frac{-xt}{1-t}. \tag{8}$$

This result is proved in a different way by Rainville [4, p. 210].

Conversely, (8) implies (7). One can also prove (7) by using only the familiar generating function

$$(1-t)^{-\alpha-1} \exp\frac{-xt}{1-t} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n.$$

We have also from (7)

$$\sum_{n=0}^{\infty} \binom{m+n}{m} L_{m+n}^{(\alpha-n)}(x)\, t^n = \sum_{r=0}^{m} \frac{(-xt)^r}{r!} L_{m-r}^{(\alpha+r)}(x) \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n.$$

Since

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt}, \tag{9}$$

the right member is equal to

$$\sum_{r=0}^{m} \frac{(-xt)^r}{r!} L_{m-r}^{(\alpha+r)}(x) \cdot (1+t)^{\alpha} e^{-xt} = L_m^{(\alpha)}(x+xt) \cdot (1+t)^{\alpha} e^{-xt}.$$

We have therefore

$$\sum_{n=0}^{\infty} \binom{m+n}{m} L_{m+n}^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt} L_m^{(\alpha)}(x(1+t)). \tag{10}$$

3. By means of (10), we can give a simple proof of the formula [1, p. 151]

$$\sum_{n=0}^{\infty} n!\, L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, t^n = \beta!\, e^{xyt}(1-yt)^{\alpha-\beta} t^{\beta} L_{\beta}^{(\alpha-\beta)}\left(-\frac{(1-xt)(1-yt)}{t}\right), \tag{11}$$

where α and β are non-negative integers (for the proof compare [4, p. 211]).

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} n!\, L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, t^n &= \sum_{n=0}^{\infty} n!\, L_n^{(\alpha-n)}(x) \sum_{r=0}^{n} \binom{\beta}{n-r} \frac{(-y)^r}{r!} t^n \\
&= \sum_{n,r=0}^{\infty} \frac{(n+r)!}{r!} L_{n+r}^{(\alpha-n-r)}(x) \binom{\beta}{n} (-y)^r t^{n+r} \\
&= \sum_{n=0}^{\infty} n! \binom{\beta}{n} t^n \sum_{r=0}^{\infty} \binom{n+r}{r} L_{n+r}^{(\alpha-n-r)}(x)\, (-yt)^r \\
&= \sum_{n=0}^{\infty} n! \binom{\beta}{n} t^n (1-yt)^{\alpha-n} e^{xyt} L_n^{(\alpha-n)}(x(1-yt)) \\
&= (1-yt)^{\alpha} e^{xyt} \sum_{n=0}^{\beta} n! \binom{\beta}{n} \left(\frac{t}{1-yt}\right)^n L_n^{(\alpha-n)}(x(1-yt)).
\end{aligned}$$

Thus to prove (11) it suffices to show that

$$\sum_{n=0}^{\beta} \frac{1}{(\beta-n)!} \left(\frac{1-yt}{t}\right)^{\beta-n} L_n^{(\alpha-n)}(x(1-yt)) = L_{\beta}^{(\alpha-\beta)}\left(-\frac{(1-xt)(1-yt)}{t}\right).$$

This is evidently contained in the "multiplication" formula

$$L_k^{(\alpha-k)}(\lambda x) = \sum_{r=0}^{k} L_r^{(\alpha-r)}(x) \frac{(1-\lambda)^{k-r} x^{k-r}}{(k-r)!},$$

which is an immediate consequence of (9).

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