

HOMOLOGY THEORY FOR LOCALLY COMPACT SPACES

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In this paper, we develop a homology theory for locally compact spaces. On compact metric spaces, our theory is equivalent to the Steenrod homology theory [8]. The main purpose of introducing it is to obtain a Poincaré duality theorem for cohomology manifolds (see Section 7). The subject matter of the present paper is essentially the same as that of [3, Chapter II]. However, more emphasis has been put on the homology theory, which is treated from a slightly different point of view. Cohomology manifolds, which were the main concern of [3, Chap. I, II], will here be discussed more briefly.

The notation will in general be that of [7]. We assume familiarity with sheaf theory [4], [7]. In particular, the following concepts and notation will be used. A *family of supports* Φ in the space X is a collection of closed subsets such that

- i) if $A, B \in \Phi$, then $A \cup B \in \Phi$, and
- ii) if $B \in \Phi$ and A is a closed subset of B , then $A \in \Phi$.

Given a sheaf \mathcal{S} on X , the symbols $\Gamma(\mathcal{S})$, $\Gamma_{\Phi}(\mathcal{S})$, $\mathcal{S}(A)$ will denote respectively the sections of \mathcal{S} , the sections of \mathcal{S} with support in Φ , and the sections of \mathcal{S} over the subspace A of X . Note that by *section* we shall always mean continuous section.

If A is a subspace of X and \mathcal{S} is a sheaf on X , we denote by $\mathcal{S}|_A$ the restriction of \mathcal{S} to A . Further, if A is locally closed, we denote by \mathcal{S}_A the sheaf on X which induces $\mathcal{S}|_A$ on A and zero on $X - A$. Recall that if A is open in X , the sequence

$$0 \rightarrow \mathcal{S}_A \rightarrow \mathcal{S} \rightarrow \mathcal{S}_{X-A} \rightarrow 0$$

is an exact sequence of sheaves on X .

A sheaf \mathcal{S} on X is *flabby* if the restriction map $\Gamma(\mathcal{S}) \rightarrow \mathcal{S}(A)$ is surjective for every open subspace A of X ; it is called *soft* if $\Gamma(\mathcal{S}) \rightarrow \mathcal{S}(A)$ is surjective for every closed subspace A of X . If Φ is a family of supports in X , then \mathcal{S} is *soft relative to Φ* if $\mathcal{S}|_A$ is soft for every $A \in \Phi$. When the family Φ is paracompactifying [7], this is equivalent to saying that $\Gamma_{\Phi}(\mathcal{S}) \rightarrow \Gamma_{\Phi}(\mathcal{S}|_A) = \mathcal{S}(A)$ is surjective for every $A \in \Phi$. Thus the word "flabby" is the translation of *flasque*, and "soft" the translation of *mou*.

Throughout this paper, we shall assume that K is a Dedekind ring given once and for all. All modules are assumed to be K -modules, and \otimes , Tor , Hom , and Ext are taken over K . All sheaves in addition to their stated properties will be assumed to be sheaves of K -modules. Finally, all topological spaces will be assumed to be Hausdorff and, from Section 2 on, locally compact.

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1. THE CANONICAL RESOLUTION OF A SHEAF

If M is a module, we denote by $F(M)$ the free module generated by the nonzero elements of M . There is a canonical surjection $F(M) \rightarrow M$, and we denote by $R(M)$ the kernel of this map. Now, since K is an integral domain, there exists a quotient field K^* of K , and $R(M)$ is canonically imbedded in $K^* \otimes F(M)$. Let \overline{M} be the quotient of $K^* \otimes F(M)$ by $R(M)$; then there exists a natural map $M \rightarrow \overline{M}$ which is injective. Moreover, \overline{M} is an injective module. Thus we have chosen functorially an imbedding of every module M in an injective module \overline{M} .

1.1. DEFINITION. If \mathcal{P} is a sheaf on X , let $I(\mathcal{P})$ be the sheaf on X such that $I(\mathcal{P})(A) = \prod_{x \in A} \overline{\mathcal{P}_x}$ for every open subset A of X , where \mathcal{P}_x is the stalk of \mathcal{P} over x .

There is a canonical injection of \mathcal{P} into $I(\mathcal{P})$, and the sheaf $I(\mathcal{P})$ is injective. This means that if \mathcal{L}' is a subsheaf of \mathcal{L} and we are given a homomorphism of \mathcal{L}' into $I(\mathcal{P})$, it can be extended to a map of \mathcal{L} into $I(\mathcal{P})$ [7, Chap. II, Section 7.1]. The condition that a sheaf be injective is stronger than that it be flabby [7, loc. cit.]. Consequently $I(\mathcal{P})$ is flabby.

1.2. LEMMA. If \mathcal{P} is an injective sheaf on X , and Φ a family of supports, $\Gamma_\Phi(\mathcal{P})$ is an injective module.

Since \mathcal{P} is injective, it is a direct summand of $I(\mathcal{P})$. Therefore it suffices to prove the lemma for $I(\mathcal{P})$. Now if $f \in \Gamma_\Phi(I(\mathcal{P}))$ and k is a nonzero element of K , there exists $g \in \Gamma_\Phi(I(\mathcal{P}))$ with exactly the same support as f such that $k \cdot g = f$. Thus $\Gamma_\Phi(I(\mathcal{P}))$ is divisible, hence injective [5, p. 134].

1.3. DEFINITION. If \mathcal{P} is a sheaf on X , define

$$\begin{aligned} \mathcal{E}^0(X; \mathcal{P}) &= I(\mathcal{P}), & \mathcal{Z}^1(X; \mathcal{P}) &= \mathcal{E}^0(X; \mathcal{P})/\mathcal{P}, \\ \mathcal{E}^q(X; \mathcal{P}) &= I(\mathcal{Z}^q(X; \mathcal{P})), & \mathcal{Z}^{q+1}(X; \mathcal{P}) &= \mathcal{E}^q(X; \mathcal{P})/\mathcal{Z}^q(X; \mathcal{P}) \quad (q \geq 1). \end{aligned}$$

Note that the sequence of sheaves

$$0 \rightarrow \mathcal{P} \rightarrow \mathcal{E}^0(X; \mathcal{P}) \rightarrow \mathcal{E}^1(X; \mathcal{P}) \rightarrow \dots$$

is exact; it will be called the *canonical injective resolution* of \mathcal{P} and will be denoted by $\mathcal{E}^*(X, \mathcal{P})$. If Φ is a family of supports on X , we put

$$C_\Phi^*(X; \mathcal{P}) = \Gamma_\Phi(\mathcal{E}^*(X; \mathcal{P}));$$

the elements of $C_\Phi^*(X; \mathcal{P})$ are the *standard cochains* of X , with coefficients in \mathcal{P} and supports in Φ .

1.4. LEMMA. If $f: \mathcal{P}' \rightarrow \mathcal{P}$ is a homomorphism of sheaves on X , there exists a canonical map $f^*: \mathcal{E}^*(X; \mathcal{P}') \rightarrow \mathcal{E}^*(X; \mathcal{P})$ such that the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{P}' & \rightarrow & \mathcal{E}^0(X; \mathcal{P}') & \rightarrow & \mathcal{E}^1(X; \mathcal{P}') & \rightarrow & \dots \\ & & \downarrow f & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{P} & \rightarrow & \mathcal{E}^0(X; \mathcal{P}) & \rightarrow & \mathcal{E}^1(X; \mathcal{P}) & \rightarrow & \dots \end{array}$$

is commutative.

This follows from the fact that the assignment $\mathcal{S} \rightarrow I(\mathcal{S})$ is functorial.

1.5. LEMMA. *If the sheaf \mathcal{S} is locally concentrated on A , and A is a locally closed subspace of X , then the natural map*

$$\mathcal{C}^*(X; \mathcal{S})|_A \rightarrow \mathcal{C}^*(A; \mathcal{S}),$$

where $\mathcal{C}^*(A; \mathcal{S}) = \mathcal{C}^*(A; \mathcal{S}|_A)$, is bijective.

Proof. For $x \in A$,

$$\mathcal{C}^0(X; \mathcal{S})(x) = \lim_{U \rightarrow x} \prod_{y \in U} \overline{\mathcal{S}}_y,$$

$$\mathcal{C}^0(A; \mathcal{S})(x) = \lim_{U \rightarrow x} \prod_{y \in A \cap U} \overline{\mathcal{S}}_y.$$

However, if U is a small enough neighborhood of x , and $y \in U - U \cap A$, we have $\mathcal{S}_y = 0$ and $\overline{\mathcal{S}}_y = 0$; this gives the desired result.

The preceding proof is an exact duplicate of a proof in [7, p. 187].

1.6. LEMMA. *If A is closed in X and \mathcal{S} is concentrated on A , then*

$$\Gamma_{\Phi}(\mathcal{C}^*(X; \mathcal{S})) = \Gamma_{\Phi \cap A}(\mathcal{C}^*(A; \mathcal{S})).$$

This lemma is just a translation of a known theorem [7, p. 188], and it is equivalent to the assertion that $C_{\Phi}^*(X; \mathcal{S}) = C_{\Phi \cap A}^*(A; \mathcal{S})$.

1.7. DEFINITION. *Suppose Φ is a family of supports on X , and Ψ a family of supports on Y ; then a map $f: X \rightarrow Y$ is proper relative to Φ and Ψ if $f^{-1}(A) \in \Phi$ for every $A \in \Psi$. It is proper if it is proper relative to the families of compact subsets.*

1.8. PROPOSITION. *If Φ, Ψ are families of supports on X and Y , and $f: X \rightarrow Y$ is a proper map relative to Φ, Ψ , then for any sheaf \mathcal{S} on Y , there is a map $f^*: C_{\Psi}^*(Y; \mathcal{S}) \rightarrow C_{\Phi}^*(X; f^*(\mathcal{S}))$ compatible with the natural map $\Gamma_{\Psi}(\mathcal{S}) \rightarrow \Gamma_{\Phi}(f^*(\mathcal{S}))$ given by $t \rightarrow t \circ f$. Moreover, the map f^* is unique up to chain homotopy.*

Since $f^*(\mathcal{C}^*(Y; \mathcal{S}))$ is a resolution of $f^*(\mathcal{S})$, the inverse image of \mathcal{S} [7, p. 145], there exists a map which is unique up to homotopy of the resolution $f^*(\mathcal{C}^*(Y; \mathcal{S}))$ into the resolution $\mathcal{C}^*(X; f^*(\mathcal{S}))$, and the result follows.

We end this section with a refinement of Theorem 3.6.1 of [7, Chap. II], which will be needed in Section 2. Recall that given a sheaf \mathcal{S} on the space X , together with a section s of \mathcal{S} and an open covering $(U_i)_{i \in I}$ of X , a partition of s subordinated to $(U_i)_{i \in I}$ is a family $(s_i)_{i \in I}$ of sections of \mathcal{S} such that the support $|s_i|$ of s_i is contained in U_i , the subspaces s_i form a locally finite family, and $\sum_{i \in I} s_i = s$ ([7], p. 155).

1.9. PROPOSITION. *Let Φ be a paracompactifying family on X , let \mathcal{S} be a Φ -soft sheaf on X , and let $(U_i)_{i \in I}$ be an open covering of X . Let $s \in \Gamma_{\Phi}(\mathcal{S})$, and let Q be an element of Φ not meeting the support $|s|$ of s . Then there exists a partition $(s_i)_{i \in I}$ of s , subordinated to $(U_i)_{i \in I}$, whose elements belong to $\Gamma_{\Phi}(\mathcal{S})$ and are zero on Q . Further, if B is a neighborhood of $|s| \cup Q$ belonging to Φ , which is contained in the union of the U_i where i runs through a subset I' of I , it may be assumed that $s_i = 0$ for $i \notin I'$.*

We assume first that X belongs to Φ , in other words, that Φ consists of all closed subsets of X , and prove the first assertion in that case. The proof follows Godement's, and we describe it briefly. We may assume (U_i) to be locally finite. Choose then a closed covering $(F_i)_{i \in I}$ of X with $F_i \subset U_i$. Consider the set E of families $(s_j)_{j \in J}$ ($J \subset I$), where s_j is a section of \mathcal{S} over X , with support in U_i , equal to zero on Q , and $\sum_{j \in J} s_j = s$ on $F_J = \bigcup_{j \in J} F_j$. The set E is nonempty, and, ordered by inclusion, is inductive. By Zorn's lemma, it is then enough to show that if $J \neq I$, $(s_j)_{j \in J}$ is not maximal in E . Let then $i \notin J$. There exists a section s_i^1 of \mathcal{S} on $Q \cup (X - U_i) \cup F_J \cup F_i$ which is zero on $Q \cup (X - U_i)$ and is equal to $s - \sum_{j \in J} s_j$ on $F_J \cup F_i$. Since \mathcal{S} is soft, it can be extended to a section s_i over X and then $(s_j)_{j \in J} \cup s_i$ is an element of E .

We now prove 1.9 in the general case. By the above, there exists on B a partition $(s_i^1)_{i \in I'}$ of $s|_B$, subordinated to the covering $(U_i \cap B)_{i \in I'}$, whose elements are zero on Q and on $(B - \text{Int } B)$. Being zero on the boundary of B , the element s_i^1 , extended by zero outside of B , is a section of \mathcal{S} , whose support is in $U_i \cap B$, hence also in Φ . We put $s_i = 0$ if $i \notin I'$, and it is clear that all our conditions are fulfilled.

2. THE DUAL OF A DIFFERENTIAL GRADED SHEAF

In the preceding section, the assumption that all spaces are locally compact was never used. However, in this section, local compactness will play a fundamental role.

2.1. *Notation.* The family of compact subsets of any space will be denoted by c . Let \mathcal{S} be a sheaf on X , and let \mathcal{T} be the presheaf on X such that

$$\mathcal{T}(U) = \text{Hom}(\Gamma_c(\mathcal{S}|U), A),$$

where A is a fixed K -module, and such that the map $\mathcal{T}(U) \rightarrow \mathcal{T}(V)$ is induced by the inclusion map $\Gamma_c(\mathcal{S}|V) \rightarrow \Gamma_c(\mathcal{S}|U)$ for $V \subset U$ open in X . Denote by \mathcal{I} the sheaf determined by the presheaf \mathcal{T} .

2.2. LEMMA. *If the sheaf \mathcal{S} is c -soft and $\{U_i\}_{i \in I}$ is an open covering of X , then the sequence*

$$\sum_{i,j \in I} \Gamma_c(\mathcal{S}|U_i \cap U_j) \xrightarrow{g} \sum_{i \in I} \Gamma_c(\mathcal{S}|U_i) \xrightarrow{h} \Gamma_c(\mathcal{S}) \rightarrow 0$$

is exact, where

- 1) $g(x_{i,j}) = y_j - y_i$, y_i being the image of $x_{i,j} \in \Gamma_c(\mathcal{S}|U_i \cap U_j)$ in $\Gamma_c(\mathcal{S}|U_i)$ and
- 2) $h(y_i)$ is the image of $y_i \in \Gamma_c(\mathcal{S}|U_i)$ in $\Gamma_c(\mathcal{S})$.

Proof. It follows immediately from 1.9 that h is surjective.

The fact that $h \circ g$ is zero is immediate. Therefore it remains to show that $\text{Ker } h \subset \text{Im } g$. Let $y \in \text{Ker } h$. We may write $y = y_i + z$, where $y_i \in \Gamma_c(\mathcal{S}|U_i)$ and $z \in \sum_{j \in J} \Gamma_c(\mathcal{S}|U_j)$, and where in turn J is some finite subset of I . Since $h(y) = 0$, the support of $h(z)$ is covered by the open sets $U_i \cap U_j$ ($j \in J$). Using a partition of unity subordinated to this covering, we can find elements $z_{ij} \in \Gamma_c(\mathcal{S}|U_i \cap U_j)$ ($j \in J$) which, viewed as elements of $\Gamma_c(\mathcal{S})$, have a sum equal to $h(z)$. Let us write $z_{ij}^{(k)}$ ($k = i, j$) for the image of z_{ij} in $\Gamma_c(\mathcal{S}|U_k)$. Then

$$g\left(\sum_{j \in J} z_{ij}\right) = -\sum_{j \in J} z_{ij}^{(i)} + \sum_{j \in J} z_{ij}^{(j)}, \quad h(z) = \sum_{j \in J} h\left(z_{ij}^{(i)}\right) = \sum_{j \in J} h\left(z_{ij}^{(j)}\right).$$

Since $h(y_i + z) = 0$, it follows immediately that $y_i + \sum_{j \in J} z_{ij}^{(i)} = 0$, hence

$$y_i + z - \sum_{j \in J} g(z_{ij}) = z - \sum_{j \in J} z_{ij}^{(j)},$$

so that it will be enough to prove that the right-hand side is contained in $\text{Im } g$. Since $z \in \sum_{j \in J} \Gamma_c(\mathcal{P}|U_j)$, our assertion follows then by a simple induction on the number of nonzero components of y .

2.3. PROPOSITION. *If \mathcal{P} is c -soft, the natural map $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is bijective. If moreover A is injective, the sheaf \mathcal{G} is flabby.*

Proof. Let $\{U_i\}_{i \in I}$ be a family of open subsets of X , and let U be their union. Suppose that $t, t' \in \mathcal{F}(U)$ and that they have the same image in each $\mathcal{F}(U_i)$; then $t = t'$; for we see by 2.2 that $\sum \Gamma_c(\mathcal{P}|U_i) \rightarrow \Gamma_c(\mathcal{P}|U) \rightarrow 0$ is exact, and therefore $0 \rightarrow \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$ is exact.

Now suppose $t_i \in \mathcal{F}(U_i)$ for $i \in I$, and t_i, t_j have the same image in $\mathcal{F}(U_i \cap U_j)$. From the exactness of

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{h^*} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{g^*} \prod_{i, j} \mathcal{F}(U_i \cap U_j)$$

it follows that there is a unique $t \in \mathcal{F}(U)$, since that $h^*(t) = \{t_i\}$. This proves the first assertion [7, p. 109].

If now A is injective, the map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ (U open in X) induced by the injection $\Gamma_c(\mathcal{P}|U) \rightarrow \Gamma_c(\mathcal{P})$ is surjective; hence \mathcal{G} is flabby.

2.4. PROPOSITION. *Let F be a closed subset of X . If \mathcal{P} is c -soft and A is injective, the restriction map $\mu: \Gamma_c(\mathcal{P}) \rightarrow \Gamma_c(\mathcal{P}|F)$ is surjective and induces a monomorphism $\nu: \text{Hom}(\Gamma_c(\mathcal{P}|F), A) \rightarrow \mathcal{F}(X)$ whose image consists of the elements of $\mathcal{F}(X)$ which have their support in F . In particular, an element $t \in \mathcal{F}(X)$ with support in F has the value zero on the elements of $\Gamma_c(\mathcal{P})$ which have their supports in $X - F$.*

Since \mathcal{P} is c -soft, μ is surjective (see Note below); therefore we have an exact sequence

$$0 \rightarrow \Gamma_c(\mathcal{P})_{X-F} \rightarrow \Gamma_c(\mathcal{P}) \xrightarrow{\mu} \Gamma_c(\mathcal{P}|F) \rightarrow 0,$$

where $\Gamma_c(\mathcal{P})_{X-F}$ means the elements of $\Gamma_c(\mathcal{P})$ with support in $X - F$. The module A being injective, this yields an exact sequence

$$0 \rightarrow \text{Hom}(\Gamma_c(\mathcal{P}|F), A) \xrightarrow{\nu} \mathcal{F}(X) \xrightarrow{\sigma} \text{Hom}(\Gamma_c(\mathcal{P})_{X-F}, A) \rightarrow 0.$$

The elements of $\text{Im } \nu$ clearly have their supports in F . Now, if $t \in \mathcal{F}(X)$ has its support in F , each $x \in X - F$ has a neighborhood V_x such that t is zero on the elements of $\Gamma_c(\mathcal{P})$ with supports in V_x . With the aid of partitions of unity, it follows

that t is zero on all elements of $\Gamma_c(\mathcal{S})$ with supports in $X - F$; therefore $t \in \text{Ker } \sigma$ and $t \in \text{Im } \nu$.

Note. In the beginning of the above proof, we have used a special case of the following elementary fact: Let F be a closed subset of X , let Φ be a paracompactifying family on X , and let \mathcal{S} be a Φ -soft sheaf. Then the restriction map

$$\mu: \Gamma_\Phi(\mathcal{S}) \rightarrow \Gamma_{\Phi|_F}(\mathcal{S}|_F)$$

is surjective. To see this, given an element t of $\Gamma_{\Phi|_F}(\mathcal{S}|_F)$ with support K , first extend its restriction to $F \cap V$ to an element of $\Gamma_{\Phi|_V}(\mathcal{S}|_V)$, where V is a neighborhood of K belonging to Φ , which is zero on $V - \text{Int } V$. Then this element, extended by zero outside V , belongs to $\mu^{-1}(t)$.

2.5. Notation and conventions. A *differential graded module* M is a sequence of modules M^q indexed on the integers, together with maps $d: M^q \rightarrow M^{q+1}$ such that $d^2 = 0$. The differential graded module M is *bounded below* if $M^q = 0$ for q less than some fixed integer n ; it is *bounded above* if $M^q = 0$ for $q > n$. We also denote M^q by M_{-q} , and we let $d: M_q \rightarrow M_{q-1}$ stand for $d: M^{-q} \rightarrow M^{-q+1}$. If M, N are differential graded modules, $\text{Hom}(M, N)_r$ is the module of maps $f: M \rightarrow N$ such that $f(M^q) \subset N^{q-r}$. Moreover, $\text{Hom}(M, N)$ is the differential graded module such that $\text{Hom}(M, N)^r = \text{Hom}(M, N)_{-r}$, and such that if $f \in \text{Hom}(M, N)_r$, then

$$(df)(x) = d(f(x)) + (-1)^{r+1} f(dx).$$

Let K^* be the quotient field of K , and let $R(K)$ be the differential graded module such that $R(K)^0 = K^*$, $R(K)^1 = K^*/K$, $R(K)^q = 0$ for $q \neq 0, 1$, and such that $d: R(K)^0 \rightarrow R(K)^1$ is the natural map $K^* \rightarrow K^*/K$. For any differential graded module M , let $D(M) = \text{Hom}(M, R(K))$. The differential graded module $D(M)$ is called the *dual of M*. By definition,

$$D(M)_r = \text{Hom}(M^r, K^*) \oplus \text{Hom}(M^{r+1}, K^*/K).$$

Moreover, since $R(K)$ is an injective resolution of K , there exists a split exact sequence

$$0 \rightarrow \text{Ext}(H^{q+1}(M), K) \rightarrow H_q(D(M)) \rightarrow \text{Hom}(H^q(M), K) \rightarrow 0.$$

Further, since $R(K)$ is an injective K -module, we see that if $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is an exact sequence of differential graded K -modules, then

$$0 \rightarrow D(M'') \rightarrow D(M) \rightarrow D(M') \rightarrow 0$$

is an exact sequence of differential graded K -modules. Notice that here we assume $df = fd$, $dg = gd$, and that f, g are of degree zero, in other words, that

$$f \in Z^0 \text{Hom}(M', M)$$

and $g \in Z^0 \text{Hom}(M, M'')$.

A *differential graded sheaf* \mathcal{A} is a sequence of sheaves \mathcal{A}^q together with maps $d: \mathcal{A}^q \rightarrow \mathcal{A}^{q+1}$ such that $d^2 = 0$. For any open set U in X and any family Φ of subsets of X , $\Gamma_\Phi(\mathcal{A}|_U)$ is a differential graded module with the component $\Gamma_\Phi(\mathcal{A}^q|_U)$ of degree q .

2.6. DEFINITION. If \mathcal{A} is any differential graded sheaf, we denote by $D(\mathcal{A})$ the differential graded sheaf determined by the presheaf $U \rightarrow D(\Gamma_c(\mathcal{A}|U))$.

2.7. PROPOSITION. If \mathcal{A} is a differential graded sheaf on X which is soft relative to the family of compact subsets, then

- 1) for U open in X , $D(\mathcal{A})(U) = D(\Gamma_c(\mathcal{A}|U))$,
- 2) $D(\mathcal{A})^q$ is flabby for every integer q , and
- 3) for every integer q , there exists a split exact sequence

$$0 \rightarrow \text{Ext}(H^{q+1}(\Gamma_c(\mathcal{A})), K) \rightarrow H_q(\Gamma(D(\mathcal{A}))) \rightarrow \text{Hom}(H^q(\Gamma_c(\mathcal{A})), K) \rightarrow 0.$$

- 4) If \mathcal{A} is injective, $D(\mathcal{A})(U)$ is torsion-free, for U open in X .

Proof. Parts 1 and 2 of the proposition follow at once from 2.3. Part 3 is an immediate corollary of the equality $\Gamma(D(\mathcal{A})) = D(\Gamma_c(\mathcal{A}))$ and of the corresponding fact about differential modules. If \mathcal{A} is injective, then $\Gamma_c(\mathcal{A}|U)$ is injective by 1.2, and (4) follows from (1). (Recall that if A is divisible, then $\text{Hom}(A, B)$ is torsion-free for any module B .)

2.8. Notation. We have already adopted the convention that if A is a module, then we also denote by A the constant sheaf with stalk A . We shall denote by \mathcal{K} the sheaf such that $\mathcal{K}(U) = \prod_{x \in U} K_x$ for U open in X , where K_x is the stalk of K over $x \in X$.

Recall that $\mathcal{K}(U)$ is a sheaf of rings, and that it is flabby, hence Φ -fine for any paracompactifying family Φ [7, Chap. II, 3.7]. As a consequence, every sheaf which is a \mathcal{K} -Module is Φ -fine, and *a fortiori* Φ -soft. Together with 2.3, this implies the following

2.9. PROPOSITION. Let A be a module, and let \mathcal{S} be a \mathcal{K} -module. Then the presheaf $U \rightarrow \text{Hom}(\Gamma_c(\mathcal{S}|U), A)$ is a sheaf which is Φ -fine for any paracompactifying family Φ .

3. HOMOLOGY THEORY

3.1. For any space X , we define $\mathcal{E}_H(X; K)$ to be the differential graded sheaf $D(\mathcal{C}^*(X; K))$. If \mathcal{S} is any sheaf on X , we denote by $\mathcal{C}_H(X; \mathcal{S})$ the sheaf $\mathcal{S} \otimes \mathcal{E}_H(X; K)$. The sheaf $\mathcal{E}_H(X; K)$ is called the *standard sheaf for homology on X* , and $\mathcal{C}_H(X; \mathcal{S})$ is called the *standard sheaf for homology with coefficients in \mathcal{S}* . If Φ is any family of supports on X , we let $C_H^\Phi(X; \mathcal{S}) = \Gamma_\Phi(\mathcal{C}_H(X; \mathcal{S}))$ and $H_n^\Phi(X; \mathcal{S}) = H_n(C_H^\Phi(X; \mathcal{S}))$. The module $H_n^\Phi(X; \mathcal{S})$ is called the *n -dimensional homology group of X with coefficients in \mathcal{S} and supports in Φ* . If Φ is the family of all closed subsets of X , we write $H_n(X; \mathcal{S})$ for $H_n^\Phi(X; \mathcal{S})$.

The *stalk* at $x \in X$ of the derived sheaf $\mathcal{H}(\mathcal{C}_H(X; K))$ is the local homology group at x , and it will be denoted $H_*^x(X; K)$. The sheaf $\mathcal{H}(\mathcal{C}_H(X; K))$ is the sheaf of local homology groups, and it will be denoted $\mathcal{H}_*(X; K)$.

For any space, the sheaf $\mathcal{C}_H(X; K)_n$ is flabby, and $\mathcal{C}_H(X; K)(U)$ is a torsion-free differential graded module for U open in X , by 2.7. Clearly, $\mathcal{C}_H(X; \mathcal{S})$ is a \mathcal{K} -module; therefore it is Φ -fine for any paracompactifying family Φ on X . For U open in X , the restriction of cross sections induces a natural homomorphism

$$j_{*XU} : H_*^\Phi(X; K) \rightarrow H_*^{\Phi \cap U}(U; K),$$

whose restriction to $H_n^\Phi(X; K)$ will be denoted by j_{nXU} .

3.2. THEOREM. *Let $f: X \rightarrow Y$ be a proper map. Then for every integer n there exists a natural map $f_*: H_n(X; K) \rightarrow H_n(Y; K)$.*

Proof. Observe that $f^*(K) = K$, where K stands for the constant sheaf with stalk K . Now by 1.8 there exists a map $f^*: C_c^*(Y; K) \rightarrow C_c^*(X; K)$ which is unique up to homotopy. This induces a map

$$Df^*: D(C_c^*(X; K)) \rightarrow D(C_c^*(Y; K)),$$

and since $D(C_c^*(X; K)) = \Gamma(\mathcal{C}_H(X; K))$, $D(C_c^*(Y; K)) = \Gamma(\mathcal{C}_H(Y; K))$, the first assertion follows.

3.3. THEOREM. (a) *For each space X and each integer i , there exists a split exact sequence*

$$0 \rightarrow \text{Ext}(H_c^{i+1}(X; K), K) \rightarrow H_i(X; K) \rightarrow \text{Hom}(H_c^i(X; K), K) \rightarrow 0$$

which is compatible with restrictions to open subsets. Hence there exists for each $x \in X$ an exact sequence

$$0 \rightarrow \text{dir lim Ext}(H_c^{i+1}(U; K), K) \rightarrow H_i^x(X; K) \rightarrow \text{dir lim Hom}(H_c^i(U; K), K) \rightarrow 0,$$

where U runs through the open neighborhoods of x .

(b) *Let \mathcal{A} be a c -soft resolution of K . Then $H_q(\Gamma(D(\mathcal{A}))) = H_q(X; K)$ and $\mathcal{H}(D(\mathcal{A})) = \mathcal{H}_*(X; K)$ ($q \in \mathbb{Z}$).*

The assertion (a) follows from 2.7, the definitions of $H_i(X; K)$, $H_i^x(X; K)$, and the fact that a direct limit of exact sequences is exact.

Since \mathcal{A} is a resolution of K , there exists a homomorphism $f: \mathcal{A} \rightarrow \mathcal{C}^*(X; K)$ which extends the identity map on K . Since each \mathcal{A}^q is c -soft,

$$H^q(\Gamma_c(\mathcal{A})) \rightarrow H^q(\Gamma_c(\mathcal{C}^*(X; K)))$$

is bijective, as follows from Theorems 4.7.1 and 4.7.2 of [7, Chap. II]. By 2.7, $H_q(\Gamma(D(\mathcal{C}^*(X; K)))) \rightarrow H_q(\Gamma(D(\mathcal{A})))$ is bijective for each q , which implies the first equality of (b). This isomorphism is obviously compatible with the restriction to an open set, and the second equality of (b) follows.

Remark. Let F be a closed subspace of X , and let \mathcal{A} be a c -soft resolution of K on X . The homomorphism derived from the natural map $D(\Gamma_c(\mathcal{A}|_F)) \rightarrow D(\Gamma_c(\mathcal{A}))$, transpose of the restriction, can be identified with $f_*: H_*(F; K) \rightarrow H_*(X; K)$, f being the inclusion of F in X . This is easily seen from the above and the commutative diagram

$$\begin{array}{ccc} \Gamma_c(\mathcal{A}) & \longrightarrow & \Gamma_c(\mathcal{A}|_F) \\ \downarrow & & \downarrow \\ \Gamma_c(\mathcal{C}^*(X; K)) & \longrightarrow & \Gamma_c(\mathcal{C}^*(X; K)|_F). \end{array}$$

3.4. THEOREM. *Let Φ be a family of supports of X . Then*

$$H_q^\Phi(X; K) = \lim_{F \in \Phi} H_q(F; K),$$

the limit being taken with respect to the homomorphisms defined by inclusion. If \mathcal{A} is a c -soft resolution of K on X , then $H_q^\Phi(X; K) = H_q(\Gamma_\Phi(D(\mathcal{A})))$ ($q \in \mathbb{Z}$).

By 2.4, $M(F) = \text{Hom}(\Gamma_c(\mathcal{C}^*(X; K)|_F), R(K))$ may be identified with the set of elements of $C_H(X; K)$ having their supports in F . Since the restriction of a c -soft sheaf to a closed subset is c -soft [7, Chap. II, 3.3.1], it follows from 3.3 that $H(M(F)) = H_*(F; K)$. The theorem follows from this, 2.4, and the remark to 3.3.

3.5. THEOREM. (a) Let $f: X \rightarrow Y$ be a map, and let Φ, Ψ be families of supports on X and Y respectively such that $f(\Phi) \subset \Psi$. Then, if the restriction of f to any $F \in \Phi$ is proper, there exists a natural homomorphism $H_q^\Phi(X; K) \rightarrow H_q^\Psi(Y; K)$ ($q \in \mathbb{Z}$). In particular, there exists a natural homomorphism $i_*: H_q^c(X; K) \rightarrow H_q^c(Y; K)$.

(b) Let F be a closed subset of X , and Ψ a family of supports on F . Then the natural homomorphism $H_q^\Psi(F; K) \rightarrow H_q^\Psi(X; K)$ is an isomorphism ($q \in \mathbb{Z}$).

The first assertion follows from 3.2 and the first part of 3.4. By 2.4, $C_H^\Psi(X; K)$ may be identified with $M_\Psi(F) = \Gamma_\Psi(D(\mathcal{C}^*(X; K)|_F))$; by 3.4, $H(M_\Psi(F)) = H_*^\Psi(F; K)$, and (b) follows.

Remark. Let f be the inclusion of a closed subspace F in X . Then

$$f_*: H_*^{\Phi|_F}(F; K) \rightarrow H_*^\Phi(X; K)$$

will often be denoted by i_{*FX} , and its restriction to $H_q^{\Phi|_F}(F; K)$ by i_{qFX} . By the above, it is induced by the inclusion of

$$M_{\Phi|_F}(F) = \Gamma_{\Phi|_F}(D(\mathcal{C}^*(X; K)|_F))$$

into $C_H(X; K)$, and $M_{\Phi|_F}(F)$ may be identified with the module of elements of $C_H^\Phi(X; K)$ having their supports in F . Applied to the inclusion $F \cap U \rightarrow U$, where U is open in X , and to the family of all closed subsets, this leads to an injective homomorphism

$$i_{FX}: (D(\mathcal{C}^*(X; K)|_F))^X \rightarrow \mathcal{C}_H(X; K),$$

hence also to a homomorphism

$$i_{*FX}: \mathcal{H}_*(F; K)^X \rightarrow \mathcal{H}_*(X; K),$$

where \mathcal{A}^X denotes the extension by zero of a sheaf \mathcal{A} on F [7, II, 2.9].

3.6. THEOREM. If Φ is a paracompactifying family on X , and

$$0 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 0$$

is an exact sequence of sheaves on X , there exists an exact homology sequence

$$\dots \rightarrow H_n^\Phi(X; \mathcal{G}') \rightarrow H_n^\Phi(X; \mathcal{G}) \rightarrow H_n^\Phi(X; \mathcal{G}'') \rightarrow H_{n-1}^\Phi(X; \mathcal{G}') \rightarrow \dots$$

Proof. Since $\mathcal{C}_H(X; K)$ is torsion-free, there exists an exact sequence of sheaves

$$0 \rightarrow \mathcal{C}_H(X; \mathcal{G}') \rightarrow \mathcal{C}_H(X; \mathcal{G}) \rightarrow \mathcal{C}_H(X; \mathcal{G}'') \rightarrow 0.$$

Since $\mathcal{C}_H(X; \mathcal{G}')$ is Φ -fine (by 3.1), the sequence

$$0 \rightarrow C_H^\Phi(X; \mathcal{P}') \rightarrow C_H^\Phi(X; \mathcal{P}) \rightarrow C_H^\Phi(X; \mathcal{P}'') \rightarrow 0$$

is exact ([7], p. 154), and the theorem follows.

3.7. THEOREM. *If B is a module, there exists an exact sequence*

$$0 \rightarrow H_n^c(X; K) \otimes B \rightarrow H_n^c(X; B) \rightarrow \text{Tor}(H_{n-1}^c(X; K), B) \rightarrow 0.$$

$\mathcal{C}_H(X; K)$ is c-fine and torsion-free by 3.1 and 2.6; therefore

$$\Gamma_c(\mathcal{C}_H(X; B)) = \Gamma_c(\mathcal{C}_H(X; K)) \otimes B$$

[1, Exp. V, lemma 3(e)], and the result follows.

3.8. THEOREM. *Let F be a closed subspace of X, and let U = X - F. Then there exists an exact sequence*

$$\cdots \rightarrow H_q(F; K) \xrightarrow{i_{qFX}} H_q(X; K) \xrightarrow{j_q^{XU}} H_q(U; K) \rightarrow H_{q-1}(F; K) \rightarrow \cdots$$

Let \mathcal{A} be a c-soft resolution of K on X. Then we have the exact sequence

$$0 \rightarrow \Gamma_c(\mathcal{A}|U) \rightarrow \Gamma_c(\mathcal{A}) \rightarrow \Gamma_c(\mathcal{A}|F) \rightarrow 0,$$

(from which the exact sequence for cohomology with compact supports is derived) as follows readily from the remark to 2.4. Since R(K) is injective, this yields an exact sequence

$$(1) \quad 0 \rightarrow D(\Gamma_c(\mathcal{A}|F)) \rightarrow D(\Gamma_c(\mathcal{A})) \rightarrow D(\Gamma_c(\mathcal{A}|U)) \rightarrow 0.$$

On account of 3.3(b), the desired homology sequence is the homology sequence of (1).

3.9. THEOREM. *Let X be the union of two open subspaces, X_1, X_2 , and let $X_{12} = X_1 \cap X_2$. Then there exists an exact sequence*

$$\cdots \rightarrow H_n^c(X_{12}; K) \xrightarrow{\alpha} H_n^c(X_1; K) + H_n^c(X_2; K) \xrightarrow{\beta} H_n^c(X; K) \xrightarrow{\partial} H_{n-1}^c(X_{12}; K) \rightarrow \cdots,$$

where α is the difference and β the sum of the inclusion homomorphisms.

The proof is straightforward; it is the same as that given in [2, Section 8] to show the existence of an exact sequence of the Mayer-Vietoris type in Φ -cohomology, and we leave it to the reader. We content ourselves to recall the definition of ∂ . Let z be a cycle of $\Gamma_c(\mathcal{C}_H(X; K))$. Since $\mathcal{C}_H(X; K)$ is c-fine by 3.1, we may write $z = z_1 + z_2$, where the support of z_1 (respectively z_2) is contained in X_1 (respectively X_2). Then $dz_1 + dz_2 = 0$, hence the support of dz_1 is equal to the support of dz_2 and is contained in X_{12} . Then the image under ∂ of the class of z is by definition the class of dz_1 .

3.10. THEOREM. *Let X be the union of two closed subspaces X_1, X_2 , and let $X_{12} = X_1 \cap X_2$. Then there exists an exact sequence*

$$\cdots \rightarrow H_n(X_{12}; K) \xrightarrow{\alpha} H_n(X_1; K) + H_n(X_2; K) \xrightarrow{\beta} H_n(X; K) \xrightarrow{\partial} H_{n-1}(X_{12}; K) \rightarrow \cdots,$$

where $\alpha(x) = i_{*X_{12}, X_1}(x) - i_{*X_{12}, X_2}(x)$, $\beta(x_1 + x_2) = i_{*X_1, X}(x_1) + i_{*X_2, X}(x_2)$.

Let M_1, M_2, M_{12} be the submodules of elements of $C_H(X; K)$ having supports in X_1, X_2 , and X_{12} respectively. Then, (see the remark to 3.4)

$$(2) \quad H(M_i) = H_*(X_i; K) \quad (i = 1, 2), \quad H(M_{12}) = H_*(X_{12}; K).$$

Let $c \in C_H(X; K)$. Then c can be written in at least one way as a sum $c = c_1 + c_2$ with $c_i \in M_i$ ($i = 1, 2$). In fact, since $C_H(X; K)$ is flabby, it has a section c_1 which is equal to c on $X_1 - X_{12}$ and to zero on $X_2 - X_{12}$. Then $c_1 \in M_1$ and $c_2 = c - c_1$ is in M_2 . Let now c be a cycle. Then $dc_1 = -dc_2$ has support in X_{12} , hence is an element of M_{12} . By definition, the image of the homology class of c is the class of dc_1 in $H(M_{12}) = H(X_{12}; K)$. It is a routine matter to verify that this definition is legitimate and that the sequence of 3.10 is exact. We leave the details to the reader.

3.11. *Remark.* In view of 3.3 (b), $H_*(X; K)$ may also be defined as the *hyper-homology invariant* of $\text{Hom}(\Gamma_c(\mathcal{A}), K)$, where \mathcal{A} is a c -soft resolution of K on X , in the sense of [5, XVII]. Here we have defined it as the homology of $\text{Hom}(\Gamma_c(\mathcal{A}), L)$, where L is an injective resolution of K . It is also the homology of $\text{Hom}(P, K)$, where P is a projective resolution of $\Gamma_c(\mathcal{A})$. As will be recalled in Section 6, this second procedure has been used by Steenrod [8] for compact metric spaces. For closed subsets of euclidean spaces, one can also resolve the first argument by taking a resolution \mathcal{A} of K on X for which $\Gamma_c(\mathcal{A})$ is a *free grating* which is homotopically c -fine (see [3, II, Section 4]).

4. KÜNNETH THEOREMS

4.1. THEOREM. *If X and Y are spaces, then, for each integer n , there exists a split exact sequence*

$$0 \rightarrow \sum_{r+s=n+1} \text{Ext}(H_c^r(X), H_s(Y)) \rightarrow H_n(X \times Y) \rightarrow \sum_{r+s=n} \text{Hom}(H_c^r(X), H_s(Y)) \rightarrow 0.$$

Proof. Let \mathcal{A} be a torsion-free c -soft resolution of K on X , \mathcal{B} a torsion-free c -soft resolution of K on Y , and \mathcal{C} an injective resolution of K on $X \times Y$. Now it is known that there exists a map $f: \Gamma_c(\mathcal{A}) \otimes \Gamma_c(\mathcal{B}) \rightarrow \Gamma_c(\mathcal{C})$ which induces a homology isomorphism. Thus, we may use $D(\Gamma_c(\mathcal{A}) \otimes \Gamma_c(\mathcal{B}))$ to compute the homology of $X \times Y$. However, $D(\Gamma_c(\mathcal{A}) \otimes \Gamma_c(\mathcal{B})) = \text{Hom}(\Gamma_c(\mathcal{A}), D(\Gamma_c(\mathcal{B})))$. Since $\Gamma_c(\mathcal{B})$ is torsion-free, $\text{Hom}(\Gamma_c(\mathcal{B}), R(K)) = D(\Gamma_c(\mathcal{B}))$ is injective [5, VII, 1.4], and the theorem follows.

4.2. COROLLARY. *If X is a compact contractible space, then $H_n(X \times Y) = H_n(Y)$ for any space Y .*

Proof. Since X is compact, we have $H_c^q(X) = H^q(X)$ for all q ; and since X is contractible, $H^q(X) = 0$ for $q \neq 0$ and $H^0(X) = K$. Now the result follows at once from the preceding theorem.

4.3. COROLLARY. *If I is the unit interval, $F: I \times X \rightarrow Y$ is a proper map, and $F_t: X \rightarrow Y$ is defined by $F_t(x) = f(t, x)$, then, for every integer n ,*

$$(F_0)_* = (F_1)_*: H_n(X) \rightarrow H_n(Y).$$

4.4. COROLLARY. *If $f, g: X \rightarrow Y$ are homotopic maps, then, for every integer n , $f_* = g_*: H_n^c(X) \rightarrow H_n^c(Y)$.*

Proof. This corollary follows at once from 4.3 and 3.4.

5. COMPARISON WITH OTHER HOMOLOGY THEORIES

Using Lemma 1.6, we see that if A is a closed subspace of X , then

$$\Gamma_c(\mathcal{C}^*(X; K_A)) = \Gamma_c(\mathcal{C}^*(A; K)).$$

Since $\Gamma_c(\mathcal{C}^*(X; K)) \rightarrow \Gamma_c(\mathcal{C}^*(X; K_A))$ is surjective, $C_H(A; K) \rightarrow C_H(X; K)$ is injective. This means that we may define $H_q(X, A)$ to be $H_q(\Gamma(\mathcal{C}_H(X; K))/\Gamma(\mathcal{C}_H(A; K)))$; and we then obtain a homology theory defined on pairs, in the sense of Eilenberg and Steenrod. Moreover, the homology groups of a point are just the usual homology groups of a point with coefficients in K . Therefore, for a finite complex X and a K -module B , $H_q(X; B)$ is the ordinary simplicial homology of X with coefficients in B . More generally, if X is a finite complex, and A is a subcomplex, then $H_q(X, A; B)$ is the simplicial homology of X modulo A with coefficients in B .

Now Eilenberg and Steenrod [6, p. 258] have formulated a notion of a continuous homology theory, and they have proved that the Čech homology theory is continuous on the category of compact pairs, and that further any homology theory which is continuous on compact pairs is naturally isomorphic with the Čech theory. For our homology theory, we see that if $L = K/M$, where M is a maximal ideal in K , then $H_*(X; L) = \text{Hom}(H_c^*(X; L), L)$; with the help of this fact, it is easy to see that our homology theory with coefficients in L is continuous on compact pairs. Therefore it coincides with the Čech theory with coefficients in L for compact spaces. This is however not in general true for compact spaces and arbitrary coefficients.

For compact spaces, the sheaf-theoretic cohomology is the same as the Čech cohomology. Further, it is possible to define the Čech cohomology by using Čech cochains [7, p. 223]. If we let $C^*(X; K)$ be the Čech cochains of X with coefficients in K , then the homology we have defined has the property that if X is compact, then $H_n(X; K) = H_n(D(\check{C}^*(X; K)))$. However, if P is a projective resolution of $\check{C}^*(X; K)$, then standard homological algebra shows that $H_n(\text{Hom}(P, K)) = H_n(D(\check{C}^*(X; K)))$. For compact separable metric spaces, Steenrod [8] defined a homology theory by first constructing Čech cochains, then choosing a particular projective resolution P of his Čech cochains and calling $H_n(\text{Hom}(P, K))$ the $(n+1)$ st homology of the space based on regular cycles. Thus for a compact separable metric space, the group $H_n(X; K)$ is simply Steenrod's $(n+1)$ st homology group.

6. LOCAL CONNECTEDNESS

6.1. DEFINITION. For any space X , the augmented q -dimensional cohomology group of X is $H^q(X; K)$ if $q > 0$, and is the cokernel of the natural map $K \rightarrow H^0(X)$ induced by the map of X into a point, for $q = 0$.

The augmented q -dimensional homology group with compact supports is $H_q^c(X; K)$ for $q > 0$, and it is the kernel of the natural map $H_0^c(X) \rightarrow K$ induced by the map of X into a point, for $q = 0$.

6.2. DEFINITION. If $f: X \rightarrow Y$ is a map, then f is cohomologically trivial in dimension q if the image of the augmented cohomology in dimension q of Y in the augmented cohomology of X is zero.

The map f is homologically trivial in dimension q if the image of the augmented q -dimensional compact homology of X in the augmented q -dimensional compact homology of Y is trivial.

6.3. DEFINITION. The space X is cohomologically locally connected (respectively, in dimension q), relative to K , if for every point $x \in X$ and every neighborhood U of x there is a neighborhood V of x in U such that the natural map $V \rightarrow U$ is cohomologically trivial (respectively, in dimension q), relative to K . We say then that X is clc_K (respectively, $q\text{-clc}_K$), and that X is clc_K^r if it is $q\text{-clc}_K$ for all $q \leq r$.

The space X is homologically locally connected (respectively, in dimension q) relative to K , if for every $x \in X$ and every neighborhood U of x there exists a neighborhood V of x in U such that $V \rightarrow U$ is homologically trivial (respectively, in dimension q), relative to K . We say then that X is hlc_K (respectively $q\text{-hlc}_K$), and that X is hlc_K^r if it is $q\text{-hlc}_K$ for all $q \leq r$.

We recall that X is locally connected if and only if it is clc_K^0 .

6.4. PROPOSITION. Let B be a K -module, and let N be an injective resolution of B of finite degree. Let \mathcal{A} be the differential graded sheaf defined on X by the presheaf $U \rightarrow \text{Hom}(\Gamma_c(\mathcal{C}_H^*(X; K)|U), N)$. Then

- (a) $\mathcal{A}(U) = \text{Hom}(\Gamma_c(\mathcal{C}_H^*(X; K)|U), N)$;
- (b) the sheaf \mathcal{A} is bounded below; \mathcal{A}^q is flabby for each q ;
- (c) if X is hlc_K^r , then $\mathcal{H}^q(\mathcal{A}) = 0$ ($q \leq r$; $q \neq 0$) and $\mathcal{H}^0(\mathcal{A}) = B$.

Proof. Parts (a) and (b) follow from 2.7. Further, by 2.7 (3), there exists, for each Y open in X , an exact sequence

$$(1) \quad 0 \rightarrow \text{Ext}(H_{q-1}^c(Y); B) \rightarrow H^q(\mathcal{A}(Y)) \rightarrow \text{Hom}(H_q^c(Y); B) \rightarrow 0$$

which is natural with respect to inclusion maps. Let now X be hlc_K^r , let $x \in X$, and let $W \subset V \subset U$ be open neighborhoods of x such that $W \rightarrow V$ and $V \rightarrow U$ are homologically trivial in dimensions $q - 1, q$ ($q \leq r$). We then consider the commutative diagram whose rows are the exact sequences (1) for $Y = U, V, W$ and whose vertical maps are defined by inclusions; simple diagram-chasing shows that

$$H^q(\mathcal{A}(U)) \rightarrow H^q(\mathcal{A}(W))$$

is trivial. This proves (c).

6.5. THEOREM. If X is hlc_K^r , and B is a K -module, there exists for $q \leq r$ an exact sequence

$$0 \rightarrow \text{Ext}(H_{q-1}^c(X), B) \rightarrow H^q(X; B) \rightarrow \text{Hom}(H_q^c(X), B) \rightarrow 0.$$

Proof. This follows at once from the preceding proposition and the standard spectral sequence of a differential graded sheaf [7, p. 176]. In particular, this last implies that $H^q(X; B) = H^q(\Gamma(\mathcal{A}))$, where \mathcal{A} is the sheaf of the preceding proposition ($q \leq r$).

6.6. THEOREM. The space X is clc_K if and only if it is hlc_K . If X is clc_K^r (respectively hlc_K^r), then it is hlc_K^{r-1} (respectively clc_K^r). If K is a field, the properties clc_K^r and hlc_K^r are equivalent.

Assume X to be clc_K (respectively clc_K^r). Let $x \in X$, and let U be a compact neighborhood of x . Choose compact neighborhoods $W \subset V \subset U$ of x such that $V \rightarrow U$ and $W \rightarrow V$ are cohomologically trivial in all dimensions (respectively, in

dimensions $q + 1, q$, with $q \leq r - 1$). Then, by the argument used in proving 6.4(c), it follows from 3.3(a) that $\overline{W} \rightarrow U$ is homologically trivial in all dimensions (respectively, in dimension q). The converse statement is proved similarly, with the use of open neighborhoods, and with reference 6.5 taking the place of 3.3(a).

6.7. LEMMA. *Let X be a connected and locally connected space, and let F be a compact subset of X . Then there exists a compact connected subset A of X such that $F \subset A$.*

This lemma is elementary and well known. It is a special case of Theorem 3.3 on p. 105 of [9].

6.8. PROPOSITION. *Let q be an integer. Assume that for each integer $m \leq q$ the space X has the following property.*

$(F_{m,K})$: *For each $x \in X$ and each neighborhood V of x , there exists a neighborhood U of x such that $\text{Im } i_{m,U,V}: H_m^c(U; K) \rightarrow H_m^c(V; K)$ is a finitely generated module.*

Let $Q \subset P$ be subspaces of X such that \overline{Q} is compact and contained in the interior of P . Then $\text{Im } i_{m,Q,P}$ is a finitely generated module for $m \leq q$.

The proof is essentially the same as the proof of Proposition 6.2 in [2], and we sketch it briefly. First, by use of suitable coverings of \overline{Q} and of induction on the number of elements of such coverings, the proof is easily reduced to that of the following statement: let $U_1, U_2 \subset P$ be open and relatively compact in X , and such that the image of $H_m^c(U_i; K) \rightarrow H_m^c(P; K)$ is finitely generated for $m \leq q, i = 1, 2$. Let $V_i \subset U_i$ be open with $\overline{V_i} \subset U_i$ ($i = 1, 2$), and let $V = V_1 \cup V_2, U = \overline{U_1} \cup U_2$. Then $\text{Im } i_{m,V,P}$ is finitely generated for $m \leq q$.

Proceeding by induction on dimension, we may assume Proposition 6.8 to be true for $q - 1$. In particular, $\text{Im } i_{m,V,P}$ ($m \leq q - 1$) and the image of

$$H_{q-1}^c(V_1 \cap V_2; K) \rightarrow H_{q-1}^c(U_1 \cap U_2; K)$$

are finitely generated. That $\text{Im } i_{q,V,P}$ is finitely generated follows then by inspection of the commutative diagram

$$(1) \quad \begin{array}{ccc} H_q^c(V_1; K) + H_q^c(V_2; K) & \rightarrow & H_q^c(U_1; K) + H_q^c(U_2; K) \\ \downarrow & & \downarrow \\ H_q^c(V; K) & \rightarrow & H_q^c(U; K) \\ \downarrow & & \downarrow \\ H_{q-1}^c(V_1 \cap V_2; K) & \rightarrow & H_{q-1}^c(U_1 \cap U_2; K) \end{array} \begin{array}{l} \searrow \\ \searrow \\ \searrow \\ \searrow \end{array} \rightarrow H_q^c(P; K),$$

where the vertical maps are parts of the exact sequences introduced in 3.9 and where the other maps are defined by inclusions.

6.9. THEOREM. *Let X be compact and hlc_K^r . Then, for each integer $q \leq r$, the modules $H_q(X; K)$ and $H^q(X; K)$ are finitely generated,*

$\text{Ext}(H^{q+1}(X; K), K)$ *is the torsion submodule of $H_q(X; K)$, and*

$\text{Ext}(H_{q-1}(X; K), K)$ *is the torsion submodule of $H^q(X; K)$.*

By 6.8, $H_q(X; K)$ is finitely generated. The other assertions follow then from 3.3(a) and 6.5.

Remark. Since the property clc_K^{r+1} implies hlc_K^r by 6.6, we could also have derived 6.9 from 3.3(a), 6.5, and from the known fact that if X is compact and clc_K^r , then $H^q(X; K)$ is finitely generated for $q \leq r$. This last statement, or rather the more general analogue of 6.8 for cohomology with closed supports (see [2, Prop. 6.3] for references), can also be given a proof similar to that of 6.8, the exact sequences of the diagram (1) being replaced by standard Mayer-Vietoris sequences, and property $(F_{m,K})$ by

$(F_{m,K}^*)$: For each $x \in X$ and each neighborhood V of x , there exists a neighborhood U of x in V such that $H^m(V; K) \rightarrow H^m(U; K)$ has a finitely generated image.

It is well known that $(F_{m,K}^*)$ is equivalent to $m\text{-clc}_K$. If K is a field, then $H_m^c(X; K) = \text{Hom}(H^m(X; K), K)$ for X compact; therefore $F_{m,K}$ and $F_{m,K}^*$ are equivalent; in view of 6.6, the condition $F_{m,K}$ for $m \leq r$ then implies the property hlc_K^r , hence is equivalent to it. We do not know whether $F_{m,K}$ for all $m \leq r$ is equivalent to hlc_K^r when K is not a field, or whether $F_{m,K}$ is equivalent to $m\text{-hlc}_K$. (According to the referee, $F_{m,K}$ is indeed equivalent to $m\text{-hlc}_K$.)

6.10. PROPOSITION. (1) *If X is locally connected, then*

$$H_{-1}(X; K) = H_{-1}^c(X; K) = 0.$$

(2) *If X is hlc_K^1 , then $H_0^c(X; K)$ is the free module generated by the components of X .*

If X is locally connected, then $H_c^0(X; K)$ is a free module (generated by the compact components of X); the equality $H_{-1}(X; K) = 0$ follows then from 3.4. If a space Y is compact and connected, then $H_c^0(Y; K) = K$, and therefore $H_{-1}^c(X; K) = 0$ follows from 3.4 and 6.7.

It is known that $H^1(X; K)$ is torsion-free for any compact space X . Let now X be hlc_K^1 , hence also clc_K^1 (6.6). By a theorem recalled in the previous remark, if $Q \subset P \subset X$ and Q is compact and lies in the interior of P , then the image of the restriction map $H^1(P; K) \rightarrow H^1(Q; K)$ is finitely generated; being also torsion-free, it is then a projective module [5, VII, 4.1]. From this it follows immediately that

$$\text{dir lim Ext}(H^1(P; K), K) = 0,$$

where P runs through the compact subsets of X . The assertion (2) is then a consequence of 3.3, 3.4, and 6.7.

Remark. If K is a field, it is clearly enough to assume in (2) that X is locally connected. We do not know whether this is true in the general case.

7. FINITE-DIMENSIONAL SPACES AND GENERALIZED MANIFOLDS

7.1. A topological space X is *finite-dimensional* over K if there exists an integer n such that for each sheaf \mathcal{S} on X and each paracompactifying family Φ , we have $H_{\Phi}^q(X; \mathcal{S}) = 0$ for $q > n$. The least such integer is called the dimension of X over K and is denoted by $\dim_K X$.

The dimension of X over K is also the least integer n such that $H_c^{n+1}(U; K) = 0$ for each open subset of X . (See [3, Chap. I, Section 5]; the proof is given there for a principal ideal domain of coefficients, but it is also valid for a Dedekind ring.)

7.2. THEOREM. Let $\dim_{\mathbb{K}} X \leq n$, and let Φ be a family of supports on X . Then

$$(a) \ H_q^{\Phi}(X; \mathbb{K}) = H_q^{\times}(X; \mathbb{K}) = 0 \quad (q \geq n + 1; \mathbf{x} \in X).$$

(b) If \mathcal{A} is a c -soft resolution of \mathbb{K} , \mathcal{S} is a torsion-free sheaf on X , and Φ is paracompactifying, then

$$H_{\star}^{\Phi}(X; \mathcal{S}) = H_{\star}(\Gamma_{\Phi}(D(\mathcal{A}) \otimes \mathcal{S})).$$

The assertion (a) follows from 3.3(a), applied to the elements of Φ , and from 3.4.

As was noticed in the proof of 3.3(b), there is a homomorphism

$$f: \mathcal{C}_H(X; \mathbb{K}) \rightarrow D(\mathcal{A})$$

which induces an isomorphism of the derived sheaves. Since \mathcal{S} is torsion-free, it follows from the Künneth rule that the homomorphism

$$f \otimes 1: \mathcal{C}_H(X; \mathbb{K}) \otimes \mathcal{S} \rightarrow D(\mathcal{A}) \otimes \mathcal{S}$$

also induces an isomorphism of the derived sheaves. Replacing in both sheaves the degrees by the opposite ones, we deduce from Theorem 4.6.2 and Section 4.13 of [7, Chap. II] that $f \otimes 1$ yields an isomorphism of $H(C_H^{\Phi}(X; \mathcal{S}))$ onto $H(\Gamma_{\Phi}(D(\mathcal{A}) \otimes \mathcal{S}))$, which proves (b).

7.3. THEOREM. Let $\dim_{\mathbb{K}} X \leq n$. Let \mathcal{S} be a sheaf on X , and let Φ be a family of supports on X . Then, if either $\mathcal{S} = \mathbb{K}$ or Φ is paracompactifying, there exists a canonical isomorphism

$$\Delta: H_n^{\Phi}(X; \mathcal{S}) \rightarrow H_{\Phi}^0(X; \mathcal{H}_n(X; \mathbb{K}) \otimes \mathcal{S}) = \Gamma_{\Phi}(\mathcal{H}_n(X; \mathbb{K}) \otimes \mathcal{S}).$$

Let us denote by \mathcal{B} the sheaf $\mathcal{C}_H(X; \mathcal{S})$ with the modified grading

$$\mathcal{B}^q = \mathcal{C}_H(X; \mathcal{S})_{n-q}.$$

Then

$$(1) \quad H^q(\Gamma_{\Phi}(\mathcal{B})) = H_{n-q}^{\Phi}(X; \mathcal{S})$$

by definition. Since $\mathcal{C}_H(X; \mathbb{K})$ is torsion-free (3.1), we have

$$\mathcal{H}^q(\mathcal{B}) = \mathcal{H}_{n-q}(X; \mathbb{K}) \otimes \mathcal{S},$$

hence in particular, by 7.2(a),

$$(2) \quad \mathcal{H}^0(\mathcal{B}) = \mathcal{H}_n(X; \mathbb{K}) \otimes \mathcal{S}; \mathcal{H}^q(\mathcal{B}) = 0 \quad (q < 0).$$

The sheaf \mathcal{B} is flabby if $\mathcal{S} = \mathbb{K}$, and it is Φ -fine if Φ is paracompactifying (3.1). By [7, II, 4.4.3], we have therefore in both cases

$$(3) \quad H_{\Phi}^p(X; \mathcal{A}) = 0 \quad (p \geq 1).$$

This last fact and [7, II, 4.6.1] imply the existence of a spectral sequence in which

$$E_2^{p,q} = H_{\Phi}^p(X; \mathcal{H}^q(\mathcal{B}))$$

and where E_∞ is the graded module associated with $H^*(\Gamma_\Phi(\mathcal{B}))$ suitably filtered. Since $\dim_K X$ is finite, the filtration of the underlying double complex may be assumed to be defined by a finite number of submodules [7, p. 195], hence we have regular convergence. By (2), we have

$$(4) \quad E_2^{p,q} = 0 \quad (p, q \geq 0), \quad E_2^{p,0} = H_\Phi^p(X; \mathcal{H}_n(X; K) \otimes \mathcal{S});$$

therefore

$$E_2^{0,0} = E_\infty^{0,0} = H^0(\Gamma_\Phi(\mathcal{B})),$$

and the theorem follows from (1).

7.4. *Remark.* If U is an open subspace of X , the restriction of cross sections to U defines a homomorphism of the above spectral sequences for X and U , and this implies the commutativity of the diagram

$$\begin{array}{ccc} H_n^\Phi(X; \mathcal{S}) & \xrightarrow{\Delta} & \Gamma_\Phi(\mathcal{H}_n(X; K) \otimes \mathcal{S}) \\ \downarrow & & \downarrow \\ H_n^\Phi \cap U(U; \mathcal{S}) & \xrightarrow{\Delta} & \Gamma_{\Phi \cap U}(\mathcal{H}_n(U; K) \otimes \mathcal{S}|_U) \end{array}$$

when either $\mathcal{S} = K$ or Φ and $\Phi \cap U$ are paracompactifying in X and U respectively.

7.5. DEFINITION. A topological space X is a homology n -manifold over K (briefly, an n -hm $_K$) if

- (1) X is finite-dimensional over K ;
- (2) the sheaf $\mathcal{H}_q(\mathcal{C}_H(X; K))$ is zero for $q \neq n$;
- (3) the sheaf $\mathcal{H}_n(\mathcal{C}_H(X; K))$ is locally isomorphic with the constant sheaf K .

If X is an n -hm $_K$, the sheaf $\mathcal{H}_n(\mathcal{C}_H(X; K))$ will be denoted by \mathcal{I} or \mathcal{I}_X , and it will be called the *orientation sheaf*. X is *orientable* if \mathcal{I}_X is isomorphic to the constant sheaf. Such an isomorphism is called an *orientation* of X .

Note that in 7.5(3) it was assumed that \mathcal{I}_X is locally isomorphic to K . Thus homology manifolds have been defined in a manner which makes them automatically locally orientable. If condition (3) is replaced by

(3)' for any point $x \in X$, the stalk of the sheaf \mathcal{I}_X is isomorphic to K , we get the condition $K - n$ of [3, II].

7.6. THEOREM. Let X be a homology n -manifold over K (or a space satisfying condition $K - n$ of 7.5); let \mathcal{I} be its orientation sheaf, Φ a family of supports, and \mathcal{S} a sheaf on X . If either $\mathcal{S} = K$ or Φ is paracompactifying, then, for all integers q , there exists a canonical isomorphism

$$\Delta: H_{n-p}(\mathcal{S}) \rightarrow H^p(X; \mathcal{I} \otimes \mathcal{S}).$$

The proof is quite similar to that of 7.2. We have now, in the notation of that proof,

$$(6) \quad \mathcal{H}^0(\mathcal{B}) = \mathcal{I} \otimes \mathcal{S}, \quad \mathcal{H}^q(\mathcal{B}) = 0 \quad (q \neq 0).$$

Therefore, in the spectral sequence considered there, we have

$$(7) \quad E_2^{p,q} = 0 \quad (q \neq 0), \quad E_2^{p,0} = H^p(X; \mathcal{F} \otimes \mathcal{S}),$$

whence

$$H_{\Phi}^p(X; \mathcal{F} \otimes \mathcal{S}) = E_2^{p,0} = E_{\infty}^{p,0} = H^p(\Gamma_{\Phi}(\mathcal{B})) = H_{n-p}^{\Phi}(X; \mathcal{S}).$$

7.7. PROPOSITION. *Let X be an n - hm_K , and \mathcal{F} its orientation sheaf. Then there exists a sheaf \mathcal{L} , locally isomorphic to K , and an isomorphism*

$$\alpha(\mathcal{L}): \mathcal{L} \otimes \mathcal{F} \rightarrow K.$$

If \mathcal{L}' is another such sheaf and $\alpha(\mathcal{L}'): \mathcal{L}' \otimes \mathcal{F} \rightarrow K$ an isomorphism, there exists a unique isomorphism $\beta: \mathcal{L} \rightarrow \mathcal{L}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{F} & \xrightarrow{\beta \otimes 1} & \mathcal{L}' \otimes \mathcal{F} \\ \alpha(\mathcal{L}') \searrow & & \swarrow \alpha(\mathcal{L}') \\ & K & \end{array}$$

is commutative.

A sheaf locally isomorphic to K may be thought of as a fibre bundle whose structural group is the group of units in K , so that our proposition is a standard fact about line bundles. We sketch the proof. Let $\{U_i\}_{i \in I}$ be an open covering such that $\mathcal{F}|_{U_i}$ is isomorphic with K on U_i . Let $f_i: \mathcal{F}|_{U_i} \rightarrow K \times U_i$ be such an isomorphism. The transition function $\theta_{ij}: U_i \cap U_j \rightarrow G$ (where G is the group of units of K) is then defined by $f_j f_i^{-1}(k, x) = (\theta_{ij}(x) \cdot k, x)$. The bundle \mathcal{L} is then defined by means of the transition functions f_{ij}^{-1} , and $\alpha(\mathcal{L})$ is the obvious map. Details are left to the reader.

7.8. DEFINITION. *Let X be an n - hm_K , and \mathcal{F} its orientation sheaf. The inverse of \mathcal{F} is a sheaf \mathcal{L} on X , locally isomorphic to K , together with an isomorphism $\alpha(\mathcal{L}): \mathcal{L} \otimes \mathcal{F} \rightarrow K$.*

Note that if X is orientable, then both \mathcal{L} and \mathcal{F} are isomorphic to K .

7.9. THEOREM. *Let X be an n - hm_K ; let \mathcal{S} be a sheaf on X , Φ a paracompactifying family of supports, \mathcal{F} the orientation sheaf, and \mathcal{L} the inverse of \mathcal{F} . Then, for each integer q , there exists an isomorphism*

$$\alpha: H_{\Phi}^q(X; \mathcal{S}) \rightarrow H_{n-q}^{\Phi}(X; \mathcal{L} \otimes \mathcal{S}).$$

Using 7.6, and the associativity and commutativity of tensor products, we have

$$H_{\Phi}^q(X; \mathcal{S}) = H_{\Phi}^q(X; \mathcal{L} \otimes \mathcal{F} \otimes \mathcal{S}) = H_{n-q}^{\Phi}(X; \mathcal{L} \otimes \mathcal{S}).$$

7.10. DEFINITION. *A topological space X is a cohomology n -manifold over K (an n - cm_K) if*

(1) *it is finite-dimensional over K ;*

(2) *for each open set U in X and each integer $q \neq n$, each point $x \in U$ has an open neighborhood V in U such that image $H_c^q(V; K) \rightarrow H_c^q(U; K)$ is zero;*

(3) *for each $x \in X$ and each neighborhood U of x , there exists an open neighborhood V of x and a free submodule A of $H_c^n(V; K)$ with a single generator, such that*

every point $y \in V$ has a fundamental system of open neighborhoods W for which image $H_c^n(W; K) \rightarrow H_c^n(U; K)$ is equal to A .

7.11. We recall that an n - cm_K is clc_K [2], [9] and has dimension n over K [3, I, 3.1]. It is said to be *orientable* if each connected component has the property assigned to V in (3). If X is connected and orientable, then $H_c^n(X; K) = K$ and for each open connected subset U , the map $H_c^n(U; K) \rightarrow H_c^n(X; K)$ is an isomorphism [2, I, 4.3]. With the above definition, cohomology manifolds are automatically locally orientable. If in (3) the requirement is made only for $y = x$, then one gets a condition paralleling (3') of 7.2. If K is a field, then the spaces satisfying (1), (2), and this weaker version of (3) are the generalized manifolds of Wilder [9] (see also [2], [3]). It is not known whether these generalized manifolds are always locally orientable.

7.12. THEOREM. *Let X be a topological space. Then the following two conditions are equivalent*

- (1) X is an (orientable) n - cm_K ;
- (2) X is an (orientable) n - hm_K and is hlc_K .

Let X be an n - cm_K . Then it is clc_K (7.11), hence locally connected, and hlc_K (6.6). That it is an n - hm_K (and orientable if it is an orientable n - cm_K) follows then easily from 3.3 and 7.10.

Let now X satisfy (2). It is then clc_K by 6.6. For U open and orientable, $H_c^q(U; K) = H_{n-q}^c(U; K)$ by 7.6, hence, if we take into account part (2) of 6.10, the condition that X be hlc_K becomes exactly the condition that X be an n - cm_K . Let now X be an orientable n - hm_K . Using 6.10, we see that for each component Y of X , $H_c^n(Y; K) = K$. This implies that X is also orientable as a cohomology manifold [2, I, 4.3].

Remark. As was recalled in 7.5, an n - cm_K is always clc_K . This is in fact a special case of a theorem relating cohomological local connectedness and local Betti numbers (see [2], [3] for more details). We do not know whether a similar theorem holds in homology, and in particular, whether a homology manifold is always hlc_K . Also, it follows from 7.1 and 7.6 that if X is an n - hm_K , then $\dim_K X \leq n + 1$. We do not know whether this bound can be brought down to n , as in the case of cohomology manifolds.

7.13. *Remarks on [2].* The discussion centering around Theorem 7.2 of [2], where the author tries to put into relations the duality theorems of [2] and [9] is marred by a mistake, pointed out by the referee of this paper, and by a misprint. Further, as was mentioned by the referee, a result of the present paper answers a question raised in [2]. We take this opportunity to rectify and complete that part of [2]. As in [2] let $h_r^*(X, K)$ be the projective limit of the groups $H^r(F; K)$, where F runs through the compact subsets of X , with respect to the usual restriction maps. If K is a field, and X is clc^r , then

$$(1) \quad h_r^* = H^r(X; K).$$

In fact, we can take a cofinal set of compact subsets F_α which are the closures of their interiors $\text{Int } F_\alpha$. Then

$$h_r^* = \lim_{\leftarrow} H_\alpha^r(F_\alpha; K) = \lim_{\leftarrow} H^r(\text{Int } F_\alpha; K).$$

Our assertion follows then from 3.4, 6.5, 6.6 and from the elementary fact that if a

vector space V is the inductive limit of vector spaces V_α , then its dual V^* is the projective limit of the spaces V_α^* .

On line 4 of [2, p. 236], one should read “inductive” instead of “projective.” Thus $h^r(X; K)$ is the *inductive* limit of the Čech homology groups $\check{H}_r(F; K)$ of the compact subsets of X . Now, K being again a field, $H^r(F; K)$ is the dual space of $\check{H}_r(F; K)$, hence, by the previous remark,

$$(2) \quad h_r^* = \text{Hom}(h^r(X; K), K).$$

Assume now that X is a connected, paracompact, orientable n - cm_K , where K is a field. Then, using property (P, Q), we see as in [2], that $h^r(X; K)$ and

$$h_r(X; K) = H_c^r(X; K)$$

have at most countable dimension. By (1), (2), 6.5, and [2, 3.2], $h_r(X; K)$ and $h^{n-r}(X; K)$ have the same dual space, namely $H^{n-r}(X; K)$, hence they are isomorphic. Conversely, if $h_r(X; K) = h^{n-r}(X; K)$, then by (1), (2), $H^{n-r}(X; K) = \text{Hom}(H_c^r(X; K), K)$. Consequently, for connected orientable paracompact n - cm_K over a field K [2, 3.2] and [2, 7.2], which is the author’s interpretation of a result of Wilder, are equivalent.

8. APPENDIX

In this appendix, we introduce and discuss some properties of a certain connected sequence of bifunctors on the category of sheaves, which yields the homology groups when the first variable is put equal to K . To start with, we assume only that our ground-ring K is an integral domain. Its field of fractions is denoted by K^* .

8.1. For any K -module B , consider B as an abelian group, imbed B in \bar{B} as described in Section 1. Let $\tilde{B} = \text{Hom}_Z(K, \bar{B})$, where Z is the ring of integers. We now have an exact sequence

$$0 \rightarrow \text{Hom}_Z(K, B) \rightarrow \text{Hom}_Z(K, \bar{B}),$$

and $B = \text{Hom}_K(K, B) \subset \text{Hom}_K(K, \bar{B})$. So we have functorially imbedded any K -module B in an injective K -module \tilde{B} .

For any sheaf \mathcal{A} , let $I(\mathcal{A})$ be the sheaf $U \rightarrow \prod_{x \in U} \tilde{\mathcal{A}}_x$. The sheaf $I(\mathcal{A})$ is injective, and as in Section 1, we may define $\mathcal{C}^*(X; \mathcal{A})$ to be the canonical injective resolution of \mathcal{A} .

8.2. Let $R(K)$ be an injective resolution of K , considered as a K -module, chosen once and for all. If \mathcal{L} is a sheaf, then $D(\mathcal{L})$ denotes the sheaf defined by the presheaf $U \rightarrow \text{Hom}(\Gamma_c(\mathcal{L}|U), R(K))$. If \mathcal{A} and \mathcal{B} are sheaves, let $\mathcal{C}_H(X; \mathcal{A}, \mathcal{B})$ be the differential graded sheaf given by the presheaf

$$U \rightarrow \text{Hom}(\Gamma_c(\mathcal{C}^*(X; \mathcal{A})|U), R(K)) \otimes \mathcal{B}(U).$$

Thus $\mathcal{C}_H(X; \mathcal{A}, \mathcal{B}) = \mathcal{C}_H(X; \mathcal{A}, K) \otimes \mathcal{B}$, and $\mathcal{C}_H(X; K, K)$ is essentially the sheaf defined in 3.1 and denoted by $\mathcal{C}_H(X; K)$.

If \mathcal{L} is c -soft, then $D(\mathcal{L})$ is flabby, $D(\mathcal{L})(U)$ is $\text{Hom}(\Gamma_c(\mathcal{L}|U), R(K))$; if \mathcal{L} is injective, $D(\mathcal{L})$ is torsion-free. The sheaf $\mathcal{C}_H(X; \mathcal{A}, K)$ is a \mathcal{K} -module, flabby, torsion-free, and Φ -fine for any paracompactifying family Φ ; and

$$\mathcal{C}_H(X; \mathcal{A}, K) = \text{Hom}(\Gamma_c(\mathcal{C}^*(X; \mathcal{A})|U), R(K)).$$

This follows from 2.3, 2.7, and 2.9 if we take into account the fact that these propositions and their proofs are valid without any change if K is an arbitrary integral domain.

8.3. Define

$$T_n(\mathcal{A}, \mathcal{B}) = H_n(\Gamma(\mathcal{C}_H(X; \mathcal{A}; \mathcal{B}))), \quad T_n^\Phi(\mathcal{A}, \mathcal{B}) = H_n(\Gamma_\Phi(\mathcal{C}_H(X; \mathcal{A}; \mathcal{B}))),$$

where Φ is a paracompactifying family on X . Then,

$$T_n(K; \mathcal{B}) = H_n(X; \mathcal{B}), \quad T_n^\Phi(K; \mathcal{B}) = H_n^\Phi(X; \mathcal{B}).$$

Note also that if \mathcal{L} is any injective resolution of \mathcal{A} , we have

$$T_n(\mathcal{A}, \mathcal{B}) = H_n(\Gamma(D(\mathcal{L}) \otimes \mathcal{B})), \quad T_n^\Phi(\mathcal{A}, \mathcal{B}) = H_n(\Gamma_\Phi(D(\mathcal{L}) \otimes \mathcal{B})).$$

In fact, there then exist homomorphisms

$$\alpha: \mathcal{C}^*(X; \mathcal{A}) \rightsquigarrow \mathcal{L}, \quad \beta: \mathcal{L} \rightarrow \mathcal{C}^*(X; \mathcal{A})$$

such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identity. They induce homomorphisms

$$\alpha': D(\mathcal{L}) \otimes \mathcal{B} \rightarrow \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}), \quad \beta': \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}) \rightarrow D(\mathcal{L}) \otimes \mathcal{B}$$

such that $\alpha' \circ \beta'$ and $\beta' \circ \alpha'$ are homotopic to the identity, and our assertion follows.

Let $0 \rightarrow \mathcal{B}' \rightarrow \mathcal{B} \rightarrow \mathcal{B}'' \rightarrow 0$ be an exact sequence of sheaves on X . Since $\mathcal{C}_H(X; \mathcal{A})$ is torsion-free (8.2), the sequence

$$0 \rightarrow \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}') \rightarrow \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}'') \rightarrow 0$$

is exact. Since $\mathcal{C}_H(X; \mathcal{A})$ is Φ -fine (8.2), the sheaf $\mathcal{C}_H(X; \mathcal{A}, \mathcal{B}')$ is Φ -soft; therefore [7, Théorème 3.5.2, p. 153] the sequence

$$0 \rightarrow \Gamma_\Phi \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}') \rightarrow \Gamma_\Phi \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}) \rightarrow \Gamma_\Phi \mathcal{C}_H(X; \mathcal{A}, \mathcal{B}'') \rightarrow 0$$

is exact and yields an exact sequence

$$(1) \quad \dots \rightarrow T_n^\Phi(\mathcal{A}, \mathcal{B}') \rightarrow T_n^\Phi(\mathcal{A}, \mathcal{B}) \rightarrow T_n^\Phi(\mathcal{A}, \mathcal{B}'') \rightarrow T_{n-1}^\Phi(\mathcal{A}, \mathcal{B}') \rightarrow \dots$$

Let now $0 \rightarrow \mathcal{A}' \rightarrow \mathcal{A} \rightarrow \mathcal{A}'' \rightarrow 0$ be an exact sequence of sheaves. It can be extended to an exact sequence $0 \rightarrow \mathcal{L}' \rightarrow \mathcal{L} \rightarrow \mathcal{C}^*(X; \mathcal{A}'') \rightarrow 0$ of injective resolutions [7, p. 261]. Then, since $R(K)$ is injective, the sequence

$$0 \rightarrow \mathcal{C}_H(X; \mathcal{A}'') \rightarrow D(\mathcal{L}) \rightarrow D(\mathcal{L}') \rightarrow 0$$

is also exact. Its elements are torsion-free (8.2), and therefore

$$0 \rightarrow \mathcal{C}_H(X; \mathcal{A}'') \otimes \mathcal{B} \rightarrow D(\mathcal{L}) \otimes \mathcal{B} \rightarrow D(\mathcal{L}') \otimes \mathcal{B} \rightarrow 0$$

is exact. The sheaf $\mathcal{C}_H(X; \mathcal{A}^n)$ is also Φ -fine (8.2); therefore its tensor product with \mathcal{B} is Φ -soft, and [7, 3.5.2, p. 153] the sequence

$$0 \rightarrow \Gamma_{\Phi}(\mathcal{C}_H(X; \mathcal{A}^n, \mathcal{B})) \rightarrow \Gamma_{\Phi}(D(\mathcal{L}) \otimes \mathcal{B}) \rightarrow \Gamma_{\Phi}(D(\mathcal{L}') \otimes \mathcal{B}) \rightarrow 0$$

is exact. In view of the remark made after the definition of $T_n(\mathcal{A}, \mathcal{B})$, this yields an exact sequence

$$(2) \quad \dots \rightarrow T_n^{\Phi}(\mathcal{A}^n, \mathcal{B}) \rightarrow T_n^{\Phi}(\mathcal{A}, \mathcal{B}) \rightarrow T_n^{\Phi}(\mathcal{A}', \mathcal{B}) \rightarrow T_{n-1}^{\Phi}(\mathcal{A}^n, \mathcal{B}) \rightarrow \dots$$

We have therefore shown that the sequence of bifunctors $T_n^{\Phi}(\mathcal{A}, \mathcal{B})$ on the category of sheaves on X , contravariant in the first variable, covariant in the second variable, is connected.

8.4. If the sheaf \mathcal{A} is injective, we may consider \mathcal{A} to be its own injective resolution. Then $T_n^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $n > 0$, if \mathcal{A} is injective. If the global dimension of the ring K is N , we may assume that $R(K)^q = 0$ for $q > N$; thus $T_n^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $n < -N$, and the functor $T_{-N}^{\Phi}(\mathcal{A}, \mathcal{B})$ is right-exact.

If K is a field, we then have $T_q^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $q < 0$, and $T_q^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $q \neq 0$ and \mathcal{A} injective. This means that in this case, for $n \geq 0$, we may consider $T_n^{\Phi}(\mathcal{A}, \mathcal{B})$ as the n -th left-derived functor of the functor $T_0^{\Phi}(\mathcal{A}, \mathcal{B})$, which is right-exact.

If K is of global dimension 1, that is, if K is a Dedekind ring which is not a field, then $T_q^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $q < -1$; and if in addition \mathcal{A} is injective, then $T_q^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for $q > 0$. Then, if we show that $T_0^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for \mathcal{A} injective, we may consider $T_n^{\Phi}(\mathcal{A}, \mathcal{B})$ as the n -th left-derived functor of the right-exact functor $T_{-1}^{\Phi}(\mathcal{A}, \mathcal{B})$. Then it remains to show that in this case $T_0^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$ for \mathcal{A} injective, which we do next.

8.5. LEMMA. *Let K be a Dedekind ring which is not a field; let Φ be a paracompactifying family, and \mathcal{A} an injective sheaf on X . Then $T_0^{\Phi}(\mathcal{A}, \mathcal{B}) = 0$.*

We have an exact sequence of presheaves which, for each open U in X is the exact sequence

$$0 \rightarrow \text{Hom}(\Gamma_c(\mathcal{A}|U), K) \rightarrow \text{Hom}(\Gamma_c(\mathcal{A}|U), K^*) \rightarrow \text{Hom}(\Gamma_c(\mathcal{A}|U), K^*/K) \\ \rightarrow \text{Ext}(\Gamma_c(\mathcal{A}|U), K) \rightarrow 0.$$

However since $\Gamma_c(\mathcal{A}|U)$ is injective, $\text{Hom}(\Gamma_c(\mathcal{A}|U), K) = 0$, and thus we have an exact sequence

$$0 \rightarrow \text{Hom}(\Gamma_c(\mathcal{A}|U), K^*) \rightarrow \text{Hom}(\Gamma_c(\mathcal{A}|U), K^*/K) \rightarrow \text{Ext}(\Gamma_c(\mathcal{A}|U), K) \rightarrow 0.$$

Since K^* and K^*/K are injective, the two left-hand presheaves are sheaves, and they are flabby (2.3). Thus (by [7, Théorème 3.1.2, p. 148]) the right-hand presheaf $U \rightarrow \text{Ext}(\Gamma_c(\mathcal{A}|U), K)$ is also a sheaf, and is flabby. Let this exact sequence of sheaves be denoted by

$$0 \rightarrow \mathcal{C}_0 \rightarrow \mathcal{C}_{-1} \rightarrow \mathcal{C} \rightarrow 0.$$

Now we may consider $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_{-1}$ as a differential graded sheaf whose differential is the natural map $\mathcal{C}_0 \rightarrow \mathcal{C}_{-1}$; and we may use \mathcal{C} instead of $\mathcal{C}_H(X; \mathcal{A}, K)$. For any sheaf \mathcal{B} , we have $H_q(\Gamma_{\Phi}(\mathcal{C} \otimes \mathcal{B})) = T_q^{\Phi}(\mathcal{A}, \mathcal{B})$, for each q . Now the stalks of the

sheaf \mathcal{C}_0 are modules over K^* , and they are injective. Therefore, for each point $x \in X$,

$$0 \rightarrow (\mathcal{C}_0)_x \rightarrow (\mathcal{C}_{-1})_x \rightarrow \mathcal{E}_x \rightarrow 0$$

is split exact, and since $(\mathcal{C}_{-1})_x$ is torsion-free, so is \mathcal{E}_x . This means that for each sheaf \mathcal{B} , the sequence

$$0 \rightarrow \mathcal{C}_0 \otimes \mathcal{B} \rightarrow \mathcal{C}_{-1} \otimes \mathcal{B} \rightarrow \mathcal{E} \otimes \mathcal{B} \rightarrow 0$$

is exact, and since \mathcal{C}_0 is flabby, that is, $\mathcal{C}_0 \otimes \mathcal{B}$ is Φ -soft for any paracompactifying family Φ , the sequence

$$0 \rightarrow \Gamma_\Phi(\mathcal{C}_0 \otimes \mathcal{B}) \rightarrow \Gamma_\Phi(\mathcal{C}_{-1} \otimes \mathcal{B}) \rightarrow \Gamma_\Phi(\mathcal{E} \otimes \mathcal{B}) \rightarrow 0$$

is exact. Consequently, if \mathcal{A} is injective, the kernel $T_0^\Phi(\mathcal{A}, \mathcal{B})$ of

$$\Gamma_\Phi(\mathcal{C}_0 \otimes \mathcal{B}) \rightarrow \Gamma_\Phi(\mathcal{C}_{-1} \otimes \mathcal{B})$$

is zero, and $T_{-1}^\Phi(\mathcal{A}, \mathcal{B}) = \Gamma_\Phi(\mathcal{E} \otimes \mathcal{B})$.

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