ESTIMATE OF A CERTAIN LEAST COMMON MULTIPLE

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Suppose that $N_1, N_2, \cdots$ are positive integers (not necessarily distinct) such that $\Sigma\, 1/N_i = 1$. If we impose the restriction that $N_i \le N$ for all i, how large can $\mathrm{lcm}[N_1, N_2, \cdots]$ be?

Clearly, by choosing $N_i = N$ $(i = 1, 2, \cdots, N)$, we obtain lcm = N; and on the other hand, the inequality $\mathrm{lcm}[N_i] \le \mathrm{lcm}[1, 2, \cdots, N] \le N!$ always holds. If we let $\Phi(N)$ denote the maximum of this lcm, then these remarks imply that $N \le \Phi(N) \le N!$. This trivial inequality leaves a wide gap in our knowledge of $\Phi(N)$, and it is our purpose to narrow the gap. It is fairly easy to strengthen the inequality to

$$C_1 N^2 \le \Phi(N) \le e^{C_2 N},$$

for example; but this improvement is slight. Our result is as follows.

THEOREM.

$$\log \Phi(N) \sim \frac{N}{\log N}.$$

Remarks. To obtain this precision, we need the prime number theorem $\pi(x) \sim x/\log x$, and its equivalent forms,

$$\log \prod_{p \le x} p \sim x, \qquad \log \mathrm{lcm}\,[1, 2, \cdots, n] \sim n.$$

Depending on the reader's taste, this may or may not be "elementary;" at any rate, our method also gives

$$\frac{C_1 N}{\log N} < \log \Phi(N) < \frac{C_2 N}{\log N},$$

using only the Tchebychev estimates of $\pi(x)$.

The proof splits into two portions:

I. If $\varepsilon > 0$ and N is large, then the conditions $N_i \le N$ and $\Sigma\, 1/N_i = 1$ imply that

$$\mathrm{lcm}[N_i] < e^{(1+3\varepsilon)N/\log N}.$$

II. If $\varepsilon > 0$ and N is large, then there exist $N_i \le N$ with $\Sigma\, 1/N_i = 1$ and

$$\mathrm{lcm}[N_i] > e^{(1-3\varepsilon)N/\log N}.$$

Proof of I. The N_i are given with the required properties. Let S be the set of primes p which divide some N_i and such that $p \ge (1 + 2\varepsilon)N/\log N$; and for p in S,

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let T_p be the set of all N_i that are divisible by the prime p. For $N_i \in T_p$, we write $N_i = a_i p$, so that

$$a_i \leq \frac{N}{p} \leq \frac{\log N}{1+2\varepsilon} \quad \text{and} \quad \operatorname{lcm}[a_i] \,\Big|\, B,$$

where

$$B = \operatorname*{lcm}_{k \leq \frac{\log N}{1+2\varepsilon}} [k] < e^{(1-\varepsilon)\log N} = N^{1-\varepsilon}.$$

We can now write

$$\sum_{N_i \in T_p} \frac{1}{N_i} = \frac{A}{Bp} \quad (A > 0), \qquad \sum_{N_i \notin T_p} \frac{1}{N_i} = \frac{C}{D} \quad (\text{with } (D, p) = 1).$$

Since $\frac{A}{Bp} + \frac{C}{D} = 1$, by hypothesis, it follows that $p \mid A$, and therefore

$$\sum_{N_i \in T_p} \frac{1}{N_i} \geq \frac{1}{B} > \frac{1}{N^{1-\varepsilon}}.$$

Finally,

$$1 \geq \sum_{p \in S} \sum_{N_i \in T_p} \frac{1}{N_i} > \frac{1}{N^{1-\varepsilon}} \sum_{p \in S} 1,$$

in other words, the number of elements of S is less than $N^{1-\varepsilon}$.

If for each prime p, p^α denotes the highest power of p for which p^α divides some N_i, then of course $p^\alpha \leq N$. We therefore obtain

$$\operatorname{lcm}[N_i] = \prod p^\alpha \leq \prod_{p \leq \frac{(1+2\varepsilon)N}{\log N}} N \cdot \prod_{p \in S} N \leq N^{\pi\left(\frac{(1+2\varepsilon)N}{\log N}\right)} \cdot N^{N^{1-\varepsilon}} < e^{(1+3\varepsilon)N/\log N}.$$

and this proves I.

To prove II, we require the following lemmas.

LEMMA 1. *If* $p_1, p_2, \cdots, p_n$ *are positive integers that are relatively prime in pairs, and* $K > (n-1)p_1 p_2 \cdots p_n$, *then the equation*

$$\frac{x_1}{p_1} + \cdots + \frac{x_n}{p_n} = \frac{K}{p_1 p_2 \cdots p_n}$$

has a solution in which all the x_i *are positive integers.*

Proof (induction on n). The result is trivial for $n = 1$; assume it for $n - 1$. Now in the above equation determine x_n $(0 < x_n \leq p_n)$ such that $K \equiv x_n(p_1 p_2 \cdots p_{n-1}) \pmod{p_n}$. If we write

$$K' = \frac{K - x_n(p_1 \cdots p_{n-1})}{p_n},$$

then

$$K' > \frac{(n-1)p_1 \cdots p_n - p_1 \cdots p_n}{p_n} = (n-2)p_1 \cdots p_{n-1},$$

and by the induction hypothesis, the equation

$$\frac{x_1}{p_2} + \cdots + \frac{x_{n-1}}{p_{n-1}} = \frac{K}{p_1 \cdots p_{n-1}}$$

has a solution, so that

$$\frac{x_1}{p_1} + \cdots + \frac{x_n}{p_n} = \frac{K'}{p_1 \cdots p_{n-1}} + \frac{x_n}{p_n} = \frac{K}{p_1 p_2 \cdots p_n}.$$

This completes the induction.

LEMMA 2. *If* $\varepsilon > 0$ *and* N *is large, then there exist* $p_1, p_2, \cdots, p_n$ *such that*

1. p_1 *is a power of* 2,
2. $p_2, p_3, \cdots$ *are odd primes, all distinct,*
3. $p_i \leq (1 + \varepsilon) \log N \quad (i = 1, 2, \cdots, n)$,
4. $N/\log^2 N \leq p_1 p_2 \cdots p_n \leq 2N/\log^2 N$.

Proof. Since $\prod'_{p \leq (1+\varepsilon)\log N} p > N$, we can certainly find an n such that

$$p_2 p_3 \cdots p_n \leq \frac{2N}{\log^2 N} < p_2 p_3 \cdots p_n p_{n+1} \qquad (p \leq (1 + \varepsilon)\log N)$$

(where $p_2, p_3, \cdots$ are the consecutive odd primes 3, 5, 7, 11 $\cdots$).

Next we can find a power of 2, call it p_1, in the interval

$$\frac{N/\log^2 N}{p_2 \cdots p_n} \leq p_1 \leq \frac{2N/\log^2 N}{p_2 \cdots p_n}$$

(in fact, if $x \geq 1/2$, the interval $[x, 2x]$ contains a power of 2).

Finally,

$$\frac{2N/\log^2 N}{p_2 \cdots p_n} < p_{n+1} \leq (1 + \varepsilon) \log N,$$

so that $p_1 \leq (1 + \varepsilon) \log N$, and the proof is complete.

Proof of II. $\varepsilon > 0$ is given, and N is large. Let $P_1, P_2, \cdots, P_K$ denote the primes in the interval $(\varepsilon N/\log N, (1 - \varepsilon)N/\log N)$, and let $p_1, p_2, \cdots, p_n$ denote the numbers given by Lemma 2. Since

$$n - 1 < \pi((1 + \varepsilon) \log N) < \varepsilon/2 \log N \qquad \text{and} \qquad p_1 p_2 \cdots p_n \leq 2N/\log^2 N,$$

it follows that $(n-1)p_1 \cdots p_n < \varepsilon N/\log N \leq P_k$ $(k = 1, 2, \cdots, K)$, and therefore Lemma 1 is applicable. This gives the existence of positive integers x_{jk} such that

$$\sum_j \frac{x_{jk}}{p_j P_k} = \frac{1}{p_1 p_2 \cdots p_n} \qquad (k = 1, 2, \cdots, K).$$

Now $K \leq \pi((1-\varepsilon)N/\log N) \leq N/\log^2 N \leq p_1 p_2 \cdots p_n$, and therefore, writing $L = p_1 p_2 \cdots p_n - K + 1$, we have

$$\sum_j \frac{L x_{j1}}{p_j P_1} + \sum_{k=2}^{K} \sum_j \frac{x_{jk}}{p_j P_k} = 1.$$

Hence, if we choose our N_i as the $p_j P_k$, with the multiplicities as indicated above, we have

$$\sum \frac{1}{N_i} = 1, \qquad N_i = p_j P_k \leq (1+\varepsilon)\log N \frac{(1-\varepsilon)N}{\log N} < N,$$

$$\operatorname{lcm}[N_i] \geq \prod_{\frac{\varepsilon N}{\log N} \leq P \leq \frac{(1-\varepsilon)N}{\log N}} P > e^{(1-3\varepsilon)N/\log N}.$$

This completes the proof.

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