A NOTE ON THE LOCAL "C" GROUPS OF GRIFFITHS

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In [1], Griffiths defined a local group $C_r(x; G)$ which extends the notion of non-local r-cut point (see [6]; Chap. 7). This group was used by Griffiths to obtain a local theorem of Hurewicz type. In addition, homology manifolds over the integers were defined, and a duality which amounts to Poincaré duality with respect to the rationals was proved.

The result of this note was used by the author, in a paper presented to the American Mathematical Society, to answer Griffiths' question whether his duality theorem can be extended to the torsion coefficients provided that the homology manifold, in his new sense, is LC^n. The result was also used to show that an LC^n_0 separable metric space is a (locally orientable) generalized manifold in the sense of Wilder if and only if it is a (locally orientable) singular homology manifold (see [4] for the definition of singular homology manifold). Subsequently, S. Mardešić has informed me that he has shown, by a modification of [3], that for lc^n_0 (in the singular sense), locally compact, Hausdorff pairs there exists an isomorphism, up through dimension n, of the singular homology groups onto the Čech homology groups (with compact carriers and an arbitrary group as coefficients). Griffiths has proved the same theorem in [2]. Thus one can now successfully by-pass the local groups of Griffiths to obtain Poincaré duality, using arbitrary coefficients and Čech homology, provided that the space is lc^∞_0 in the singular sense as follows:

A locally compact, finite-dimensional, lc^∞_0 *(in the singular sense) space* X *is a locally orientable, singular homology* n-*manifold with respect to an arbitrary coefficient group* G *if and only if*

(a) *for each* $x \in X$, $\check{H}_p(X, X - x) \approx \begin{cases} 0 & (p \neq n), \\ G & (p = n); \end{cases}$

(b) *for each* $x \in X$, *there exists an open* U_x *such that*

$$i_*: \check{H}_n(X, X - \bar{U}) \to \check{H}_n(X, X - y)$$

is an isomorphism onto for all $y \in U$.

($\check{H}$ denotes the Čech homology with compact carriers.)

Hence, the duality theorems [4] for lc^∞_0, locally orientable, singular homology n-manifolds can be stated either in terms of the Čech homology with compact carriers or in terms of the singular homology.

As an interesting consequence it follows that *an* lc^∞_0 *(in the singular sense), locally orientable generalized manifold in the sense of Wilder is locally separable metric and, furthermore, that it is metrizable if and only if it is paracompact.* This is a direct consequence of [4, 3.9, Remark 1].

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These remarks, which follow immediately from material in [4] and Mardešić's result, now not only simplify and strengthen our previous comparison of singular homology manifolds with generalized manifolds and our answer to Griffiths question, but also replace the local groups of Griffiths by those cited in (a). Thus the theorem of this note is no longer needed for obtaining a Poincaré duality theorem for integers (Čech homology with integers as coefficients). Nevertheless, there is a close relationship between the local groups of Griffiths and the local group cited in (a). Furthermore, Griffiths' local groups are of independent interest because of their connection with the local Hurewicz theorem; therefore we shall prove the following proposition.

THEOREM. *Let* X *be* r-lc *and* $(r+1)$-lc *at* $x \in X$. *Then the local group* $C_r(x; G)$ *exists at* x, *is stable, and is naturally isomorphic to* $H_{r+1}(X, X - x)$. (In a conversation with Griffiths, I learned that J. H. C. Whitehead has independently come upon a part of this result.)

By $U \Subset V$ we shall mean that $\bar{U} \subset V$; and if $i: U \subset V$ then, following Griffiths, we shall denote $(i^* H_r(U)) \subset H_r(V)$ by $H_r(U \mid V)$. The letters U, V and W will always denote open sets in a locally compact Hausdorff space X.

We shall fix the notation, in that H will denote an augmented exact homology theory defined on the category of locally compact Hausdorff pairs. (We shall make no use of the homotopy axiom.) In particular, H may denote the augmented singular homology theory with arbitrary coefficients, or the augmented Čech homology with compact carriers, with coefficients in a field.

Definition 1. We recall that X has a *local group* $C_r(x)$ (or $C_r(x; G)$, if we choose to emphasize the coefficient group) *at* x if there exists a neighborhood P of x such that $x \in U \Subset P$ implies that there exists a Q such that, if $x \in W \Subset Q$, then $H_r(\bar{W} - x \mid \bar{U} - x) \approx C_r(x)$.

Proof of the theorem. By the r-lc and $(r+1)$-lc hypotheses, there exists an open set P (P may be chosen with compact closure, since X is locally compact) such that $H_r(\bar{P} \mid X) = H_{r+1}(\bar{P} \mid X) = 0$. Take any open Q and U with $x \in Q \Subset U \Subset P$ such that $H_{r+1}(\bar{Q} \mid \bar{U}) = H_r(\bar{Q} \mid \bar{U}) = 0$. Let W be any open set such that $x \in W \Subset Q$. Consider the commutative diagram

$$\begin{array}{ccccccc}
H_{r+1}(\bar{W}) & \xrightarrow{j_w} & H_{r+1}(\bar{W}, \bar{W} - x) & \xrightarrow{d_w} & H_r(\bar{W} - x) & \xrightarrow{i_w} & H_r(\bar{W}) \longrightarrow \\
\downarrow i_{uw} & & \downarrow l_{uw} & & \downarrow i_{uw} & & \downarrow i_{uw} \\
H_{r+1}(\bar{U}) & \xrightarrow{j_u} & H_{r+1}(\bar{U}, \bar{U} - x) & \xrightarrow{d_u} & H_r(\bar{U} - x) & \xrightarrow{i_u} & H_r(\bar{U}) \longrightarrow \\
\downarrow i_{xu} & & \downarrow l_{xu} & & \downarrow i_{xu} & & \downarrow i_{xu} \\
H_{r+1}(X) & \xrightarrow{j_x} & H_{r+1}(X, X - x) & \xrightarrow{d_x} & H_r(X - x) & \xrightarrow{i_x} & H_r(X) \longrightarrow .
\end{array}$$

All maps are induced by inclusions, except for the three boundary maps. The homomorphisms i_{uw} and i_{xu} are trivial, by the r-lc and $(r+1)$-lc assumptions. The homomorphisms l_{uw} and l_{xu} are isomorphisms onto, by excision. We shall show that $\theta: H_r(\bar{W}, \bar{W} - x) \to H_r(\bar{W} - x \mid \bar{U} - x)$ is an isomorphism onto ($\theta = i_{uw} d_w$). With simple diagram-chasing, it follows that the map d_w has kernel 0, and hence the condition $(r+1)$-lc implies that θ has kernel 0. Again, simple diagram-chasing shows that if X is r-lc at x, then θ is onto.

We shall now define what we mean by stable, and show that $C_r(x)$ is stable under the assumptions of the theorem. The notion of stability was not used by Griffiths, and its lack hampered the usefulness of the "C" groups.

Definition 2. A local group $C_r(x)$ at $x \in X$, if it exists, is said to be *stable*, if whenever $x \in W' \Subset W$ in Definition 1, the inclusion

$$i_*(H_r(\bar{W}' - u \mid \bar{U} - x)) \subset H_r(\bar{W} - x \mid \bar{U} - x)$$

is onto.

Clearly, $C_r(x)$ exists. We take W' such that $x \in W' \Subset W$, and we consider the commutative diagram:

$$\begin{array}{ccc}
H_{r+1}(\bar{W}, \bar{W}' - x) & \xrightarrow{d_{w'}} & H_r(\bar{W}' - x) \\
\downarrow l_{ww'} & & \downarrow i_{ww'} \\
H_{r+1}(\bar{W}, \bar{W} - x) & \xrightarrow{d_w} & H_r(W - x) \\
\downarrow l_{uw} & & \downarrow i_{uw} \\
H_{r+1}(\bar{U}, \bar{U} - x) & \xrightarrow{d_u} & H_r(\bar{U} - x).
\end{array}$$

Let $i_*: H_r(\bar{W}' - x \mid \bar{U} - x) \to H_r(\bar{W} - x \mid \bar{U} - x)$. The map i_* has kernel 0, since the first group is a subgroup of the second. That i_* is onto follows from the fact that $l_{ww'}$ and θ are isomorphisms. This completes the proof.

Let $x \in X$, and let $\{U_x\}$ be a neighborhood base of open sets at x. The collection $\{U_x\}$ forms an inverse system of sets, directed by inclusion.

COROLLARY. *Let* X *be* r-lc *and* (r + 1)-lc *at* $x \in X$. *Then, there exists an isomorphism* ψ *of* $H_{r+1}(X, X - x)$ *onto the inverse limit of* $\{H_r(\bar{U}_x - x)\}$.

Proof. We abbreviate Inverse limit $\{H_r(\bar{U}_x - x)\}$ to Inv lim $H_r(\bar{U} - x)$. We may assume that all $U_x \Subset P$, where P denotes the neighborhood in the definition of $C_r(x)$.

Let $a \in H_{r+1}(X, X - x)$. Following the notations used in the proof of the theorem, we let $d_u l_{xu}^{-1}(a) = a_u \in H_r(\bar{U} - x)$. The collection $\{a_u\}$ defines an element of Inv lim $H_r(\bar{U} - x)$. For if U_x and V_x are given, there exists a W_x such that $W_x \Subset U_x \Subset V_x$ and $i_{uw}(a_w) = a_u$, $i_{vw}(a_w) = a_v$. We define $\psi(a) = a_u$. Clearly, ψ has kernel 0.

We claim that ψ is also onto. Let $b \in$ Inv lim $H_r(\bar{U} - x)$. We denote the "co-ordinate" of b on $H_r(\bar{U} - x)$ by b_u. Let $x \in V \Subset U$ be such that

$$\theta_{uv}: H_{r+1}(\bar{V}, \bar{V} - x) \to H_r(\bar{V} - x \mid U - x)$$

is an isomorphism onto. There exists a unique $a \in H_{r+1}(X, X - x)$ such that $\theta_{uv}(l_{xv}^{-1}(a)) = a_u = b_u$. We shall show that $\psi(a) = b$.

Consider, for convenience, the commutative diagram

$$\begin{array}{ccccc}
 & \nearrow^{l_{ux}} & H_{r+1}(\bar{U}, \bar{U} - x) & \xrightarrow{d_u} & H_r(\bar{U} - x) \\
 & & \uparrow l_{uv} & & \uparrow i_{uv} \\
H_{r+1}(X, X - x) & \xrightarrow{l_{vx}} & H_{r+1}(\bar{V}, \bar{V} - x) & \xrightarrow{d_v} & H_r(\bar{V} - x) \\
 & & \uparrow l_{vw} & & \uparrow i_{vw} \\
 & \searrow_{l_{wx}} & H_{r+1}(\bar{W}, \bar{W} - x) & \xrightarrow{d_w} & H_r(\bar{W} - x),
\end{array}$$

where W has been chosen so that $\theta_{VW}: H_{r+1}(\bar{W}, \bar{W} - x) \rightarrow H_r(\bar{W} - x \mid \bar{V} - x)$ is an isomorphism onto. There exists a unique $c \in H_{r+1}(X, X - x)$ such that $\theta_{VW} 1_{WX}^{-1}(c) = b_V$. By virtue of the stability of $C_r(x)$, $i_{VW}\theta_{VW} = \theta_{UW}$; hence $c = a$. Thus, given b and a coordinate b_u, we have found a unique $a \in H_{r+1}(X, X - x)$ such that $a_u = b_u$. Moreover, if $x \in W \Subset U$, then $a_W = b_W$.

Suppose now that we had started with a neighborhood V and a "coordinate" b_V. We can find $c \in H_{r+1}(X, X - x)$ such that $c_V = b_V$. If $x \in W \Subset V$, then $c_W = b_W$. We may choose W sufficiently small so that $W \Subset V \cap U$, and this will imply that $c_W = a_W$, hence that $c = a$. Therefore, ψ is onto.

REMARKS

1. If $H_{r+1}(X, X - x) = 0$ and X is r-lc at x, then $C_r(x) = 0$.

2. If $C_r(x) = 0$ and X is (r + 1)-lc at x, then $H_{r+1}(X, X - x) = 0$.

3. There exist examples to show that the condition (r + 1)-lc is necessary for the kernel θ to be 0; and the condition r-lc is necessary for the image of θ to be onto.

4. Two examples:

a) Let X be the sequence of line segments of the plane defined by $y_n(x) = 1/n$, for all positive integers n, together with the limiting segment $y_0(x) = 0$. (We take $-1 \leq x \leq 1$). Let $x = (0, 0)$. The space X, with the induced topology, is not 0-lc at x, but is 1-lc at x. The group $H_1(X, X - x; G)$ is isomorphic to G, in singular homology, but $C_0(x; G)$ is isomorphic to $\Sigma_{n=0}^{\infty} G$ (direct sum) and is not stable.

b) Let X be the subset of the plane consisting of the union of the triangles

$$X_n = \{(x, y): y = (1/n)x,\ 0 \leq x \leq 1/n\} \cup \{(x, y): x = 1/n,\ 0 \leq y \leq 1/n^2\}$$
$$\cup \{(x, y): y = 0,\ 0 \leq x \leq 1/n\}\ .$$

The space X with the induced topology is 0-lc at (0, 0), but fails to be 1-lc at (0, 0). Furthermore, $C_0(x)$ does not exist at (0, 0). The group $H_1(X, X - x; G)$ is infinitely generated.

5. In the case of the singular homology theory, we need not use the closures of the open sets W and U to define the local group $C_r(x)$. Consequently, we may still prove the theorem, since we are still able to establish an isomorphism

$$i_*: H_r(U, U - x) \rightarrow H_r(X, X - x)$$

by excision.

6. If V is an open neighborhood of $x \in X$, with compact closure, then the inclusion $i_*: H_r(V, V - x) \rightarrow H_r(X, X - x)$ is an isomorphism onto when we use the Kolmogoroff or projective homology [5] with compact supports and an arbitrary coefficient group. Thus, by using the open sets W and U instead of the closures in the definition of local "C" groups, we are able to deduce the theorem and corollary for the Kolmogoroff or projective homology with an arbitrary coefficient group.

When the coefficient group is a field, the Kolmogoroff and projective homology on a locally compact open pair (X, U) agrees with the Čech homology with compact carriers. Hence the analogue for Čech homology of the statement in Remark 5 (concerning singular homology) is true.

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