FINITENESS OF CLASS NUMBERS OF REPRESENTATIONS OF ALGEBRAS OVER FUNCTION FIELDS

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1. THE THEOREM

The purpose of this note is to establish a function-field analogue of Zassenhaus's Theorem [4] concerning the finiteness of class numbers of representations of semi-simple algebras over number fields.

Let $\mathfrak{o}$ be an integral domain with quotient field k, and let A be a finite-dimensional k-algebra. An *$\mathfrak{o}$-order* in A is defined to be an $\mathfrak{o}$-subalgebra $\mathfrak{O}$ which is finitely generated as an $\mathfrak{o}$-submodule, and such that $k\mathfrak{O} = A$. All A-modules considered will be right unitary A-modules of finite dimension over k. The $\mathfrak{O}$-submodules of an A-module V which are finitely generated as $\mathfrak{o}$-modules are called *$\mathfrak{O}$-representation submodules.* The *class number (relative to $\mathfrak{O}$)* of V is defined to be the number of nonisomorphic $\mathfrak{O}$-representation submodules M of V which *generate* V in the sense that kM = V.

If $\mathfrak{o}$ is a principal ideal domain, $\mathfrak{O}$-representation submodules have free $\mathfrak{o}$-module bases, and the definition of class number can be formulated in the obvious way in terms of matrix representations.

The $\mathfrak{O}$-representation submodules M of A generating A are the so-called *right $\mathfrak{O}$-ideals* of A. Two such $\mathfrak{O}$-ideals M and N are isomorphic if and only if there is a unit x of A such that xM = N, that is, if and only if M and N are *equivalent* $\mathfrak{O}$-ideals (see [4]). Hence, the class number of A relative to $\mathfrak{O}$, A being considered as an A-module, coincides with the *right ideal class number of* A relative to $\mathfrak{O}$ which is defined as the number of inequivalent right $\mathfrak{O}$-ideals of A. If A is semi-simple, or more generally, if A is a Frobenius algebra, this is equal to the left ideal class number of A [3], and can be called simply the *ideal class number of* A *(relative to $\mathfrak{O}$).*

Artin [1] extended a classical result for number fields by proving that a semi-simple rational algebra has finite ideal class number relative to any maximal order. Zassenhaus [4] gave a new proof, establishing the more general result which is case (1) of the following theorem.

THEOREM. *Let $\mathfrak{o}$ be either*

(1) *the ring of integers in a finite algebraic number field, or*

(2) *the integral closure of* F[X] *in a finite extension field* k *of* F(X), *where* F *is a finite field and* X *is transcendental over* F. *Then, if $\mathfrak{O}$ is an $\mathfrak{o}$-order in a* k-*algebra* A, *every completely reducible* A-*module has finite class number relative to $\mathfrak{O}$.*

In this note we establish the result for case (2). Once it has been shown in Section 3 that a division algebra has finite ideal class number, it is possible to apply the rest of Zassenhaus's argument with only verbal changes. In Section 4 we give an alternative proof, based on a lemma from [2], of the extension from the case of irreducible A-modules to that of completely reducible A-modules.

Received June 17, 1959.

It is to be noted that orders are not assumed to be maximal in the theorem, and that case (2) implies the existence of nonseparable semi-simple algebras for which every A-module has finite class number. In fact, finite nonseparable field extensions of k have this property. Simple examples show that in both cases the class number of an indecomposable right ideal component of A can be infinite if A is not semi-simple. For instance, take A to be the F(X)-algebra of matrices $U = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ with x, y and z in F(X), and let $\mathfrak{O}$ be the F[X]-order in A consisting of those U with x, y and z in F[X]. For a given monic polynomial n in F[X], mapping x onto $\begin{pmatrix} x & ny \\ 0 & z \end{pmatrix}$, defines a representation of $\mathfrak{O}$ in F[X], and the corresponding $\mathfrak{O}$-representation modules M_n are nonisomorphic for different n.

2. PRELIMINARY LEMMAS

Since in case (2) of the Theorem, $\mathfrak{o}$ is finitely generated as an F[X]-module, it is reasonably clear that we have only to consider the case $\mathfrak{o}$ = F[X], k = F(X). Therefore, *in the rest of this note,* $\mathfrak{o}$ *will denote a domain* F[X], *where* F *is the finite field of* q *elements and* X *is transcendental over* F; *and* k *will denote its quotient field* F(X). Then, of course, $\mathfrak{o}$ is a principal ideal domain with finite residue class rings. We need the following two lemmas for modules over $\mathfrak{o}$.

LEMMA 1. *Let* M *be an* $\mathfrak{o}$*-submodule of a finitely generated torsion-free* $\mathfrak{o}$*-module* N *such that* $M \subseteq N$ *and* rank M = rank N. *Let* $u_1, \cdots, u_n$ *and* $v_1, \cdots, v_n$ *be* $\mathfrak{o}$*-module bases of* M *and* N, *respectively, with* $u_i = \Sigma c_{ij} v_j$, $c_{ij} \in \mathfrak{o}$. *Then* $N:M = \mathfrak{o} : (\Delta)$, *and* $\Delta N \subseteq M$, *where* $\Delta = \det(c_{ij})$.

(Here N:M denotes the group-theoretic index of M in N.) This well-known lemma is an easy consequence of the structure theory of modules over principal ideal domains, and it holds for any principal ideal domain $\mathfrak{o}$.

LEMMA 2. *Let* M *be a finitely generated* $\mathfrak{o}$*-submodule of a vector space* V *over* k. *Then there exist only finitely many* $\mathfrak{o}$*-submodules* N *of* V *such that* $M \subseteq N$, rank M = rank N *and* $N:M = \mu$.

Proof. Using the notation of Lemma 1 relative to M and N, we have

$$N:M = \mathfrak{o} : (\Delta) = \mu;$$

therefore $q^{\deg \Delta} = \mu$. Hence, there are only finitely many Δ to consider. Moreover, $\Delta M \subseteq \Delta N \subseteq M$, and $M:\Delta M$ is finite as a power of μ. Hence, there are only finitely many modules between M and ΔM. Since the mapping of $u \in V$ onto Δu is an automorphism of V, it follows that only finitely many N can exist.

This lemma is the function-field analogue of [4; Lemma 1]. It is valid for any principal ideal domain $\mathfrak{o}$ having finite residue class rings, since for any such $\mathfrak{o}$, the number of a in $\mathfrak{o}$ with $\mathfrak{o} : (a) = \mu$ is finite.

3. THE CASE OF A DIVISION RING

As always, we assume $\mathfrak{o}$ = F[X] and k = F(X). For the proof of the theorem, we shall first assume that A is a division ring and prove that A has finite ideal class number relative to $\mathfrak{O}$. Each right $\mathfrak{O}$-ideal of A is equivalent to a nonzero right ideal of $\mathfrak{O}$. Let $\mathfrak{a} \neq 0$ be a right ideal of $\mathfrak{O}$, and let $\omega_1, \cdots, \omega_n$ be a free $\mathfrak{o}$-module basis of $\mathfrak{O}$. By Lemma 1, $\mathfrak{O} : \mathfrak{a}$ is a power q^{μ} of q. Write $\mu = n\lambda + r$ $(0 \leq r < n)$. The number of elements in $\mathfrak{O}$ of the form $\Sigma t_i \omega_i$ with $t_i \in F[X]$ and $-\infty \leq \deg t_i \leq \lambda$ is $q^{(\lambda+1)n} > q^{\mu}$; therefore, there exist two distinct elements of this form which are congruent modulo $\mathfrak{a}$. It follows that $\mathfrak{a}$ contains a nonzero element $\alpha = \Sigma s_i \omega_i$ $(-\infty \leq \deg s_i \leq \lambda)$. By Lemma 1, $\mathfrak{O} : \alpha\, \mathfrak{O} = q^{\deg N(\alpha)}$, where $N(\alpha)$ is the determinant of the image of α under the regular representation of $\mathfrak{O}$. If $\omega_i \omega_j = \Sigma c_{ijk} \omega_k$, we have $\omega_i \alpha = \Sigma s_j \omega_i \omega_j = \Sigma s_j c_{ijk} \omega_k$, and therefore $N(\alpha) = \det (\Sigma s_j c_{ijk})$. Since the determinant is a homogeneous function of degree n in the s_j, and $\deg s_j \leq \lambda$, we see that $\deg N(\alpha) \leq n\lambda + c \leq \mu + c$, where c is a constant independent of $\mathfrak{a}$ and α. Thus $\mathfrak{O} : \alpha\, \mathfrak{O} \leq q^{\mu+c} = (\mathfrak{O} : \mathfrak{a}) q^c$; hence

$$\alpha^{-1} \mathfrak{a} : \mathfrak{O} = \mathfrak{a} : \alpha \mathfrak{O} = \frac{\mathfrak{O} : \alpha \mathfrak{O}}{\mathfrak{O} : \mathfrak{a}} \leq q^c .$$

This proves that every right ideal of $\mathfrak{O}$ is equivalent to a right $\mathfrak{O}$-ideal $\mathfrak{b}$ such that $\mathfrak{b} \supseteq \mathfrak{O}$ and $\mathfrak{b} : \mathfrak{O} \leq q^c$. It is a consequence of Lemma 2 that only finitely many such $\mathfrak{b}$ exist, and therefore A has finite ideal class number.

4. COMPLETION OF THE PROOF

To show that an irreducible A-module has finite class number, we may assume that A is a full matrix ring over a division ring, and reduce the problem by means of Zassenhaus's argument [4; pp. 282-283] to the case of the division ring treated in Section 3. We omit the details.

The extension from the case of irreducible A-modules to that of completely reducible A-modules can also be carried out by the method of [4]. As an alternative we note that we may assume that A is semi-simple, and carry out this extension using

LEMMA 3. *Given* $\mathfrak{O}$*-representation modules* M *and* N, *the number of inequivalent* $(\mathfrak{O}, \mathfrak{o})$*-exact sequences* $0 \to N \to H \to M \to 0$ *is finite.*

Proof. According to [2], the inequivalent $(\mathfrak{O}, \mathfrak{o})$-exact sequences

$$0 \to N \to H \to M \to 0$$

are in one-to-one correspondence with the elements of $\mathrm{Ext}^1_{(\mathfrak{O},\mathfrak{o})}(M, N)$. Since A is semi-simple,

$$\mathrm{Ext}^1_{(A,k)}(M_k, N_k) = H^1(A; M_k, N_k) = 0, \qquad M_k = M \otimes_{\mathfrak{o}} k, \quad N_k = N \otimes_{\mathfrak{o}} k .$$

Since $\mathfrak{O}$ is a finitely generated $\mathfrak{o}$-module, it follows that every element of $\mathrm{Ext}^1_{(\mathfrak{O},\mathfrak{o})}(M, N)$ has a nonzero annihilator in $\mathfrak{o}$. Because M and N are finitely generated $\mathfrak{o}$-modules, so is $\mathrm{Ext}^1_{(\mathfrak{O},\mathfrak{o})}(M, N)$. Since $\mathfrak{o}$ has finite residue class rings, it follows that this group is finite.

Now we prove the finiteness of class numbers of A-modules by induction. If the A-module V is irreducible, the result follows from the first paragraph of this section. Otherwise, V contains an A-submodule $X \neq 0$, V. Let H be an $\mathfrak{O}$-representation submodule generating V; then $M = H \cap X$ is an $\mathfrak{O}$-representation submodule generating X, and H/M is isomorphic with an $\mathfrak{O}$-representation submodule generating V/X. By the induction hypothesis, only finitely many nonisomorphic M's and N's can occur. Further, the sequence $0 \to M \to H \to H/M \to 0$ is $(\mathfrak{O}, \mathfrak{o})$-exact, since $\mathfrak{p}$ is a principal ideal domain, H is finitely generated, and H/M is torsion free. It therefore follows by Lemma 3 that the number of nonisomorphic $\mathfrak{O}$-representation submodules generating V is finite.

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