MINKOWSKI'S AND RELATED PROBLEMS FOR CONVEX SURFACES WITH BOUNDARIES

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1. INTRODUCTION

The theory of convex bodies establishes uniqueness for general closed convex hypersurfaces in E^n, when an elementary symmetric function $R_1 R_2 \cdots R_m + \cdots$ $(1 \leq m \leq n - 1)$ of the principal radii of curvature, or a suitable generalization of it in terms of set functions, is given as function of the normal. The method was initiated by Minkowski [8]. The general problem was solved independently by A. D. Alexandrov [1] and Fenchel and Jessen [5]. A convenient source in book form is provided by [3]. For surfaces of class C^2 in E^3 and given $R_1 R_2 < \infty$ Chern [4] gave a proof avoiding the Brunn-Minkowski Theory.

Also without using this theory, Hsiung [6, 7] proves the corresponding uniqueness theorems for smooth surfaces of positive curvature with a boundary in the cases: general n, m = 1, and n = 3, m = 2, following Chern's method in the latter case.

(Hsiung's historical remarks regarding the other methods cannot pass unchallenged. The decisive papers of Alexandrov, Fenchel and Jessen are not mentioned at all. Instead we find: uniqueness for closed surfaces ". . . was established by Minkowski and proved several decades later by Lewy for analytic surfaces . . . ." Lewy, as well as other cited authors, are concerned with the existence of smooth surfaces with smooth data and not with uniqueness.)

It is the purpose of this note to show that *the theory of convex bodies is applicable to surfaces with boundaries, just because it is not restricted to smooth surfaces.* In Theorem I we establish uniqueness under very wide conditions for surfaces in E^n with a given boundary and any given (generalized) $R_1 R_2 \cdots R_m + \cdots$. The completely general case would have required a modification of the theory, which will not be discussed here.

In addition we give *a uniqueness theorem and an existence theorem for convex caps* (Theorems II and III). A cap is a convex hypersurface with a closed hyperplane boundary, such that the normal projection of the surface on the plane of the boundary lies inside or on the latter. The interest in caps derives from their central role in the investigations of Pogorelov; see [3].

2. THE AREA FUNCTIONS

We use the term convex (hyper)surface as in [3] for a connected relative open subset K of a complete convex hypersurface K* in E^n. A normal to K is the unit normal to a supporting plane of K pointing into the exterior of the convex set bounded by K*. There is an ambiguity only in the trivial case, disregarded here, of a non-closed K lying in a hyperplane. The spherical image of K consists of the endpoints on the unit sphere Σ of the unit vectors beginning at the center of Σ and parallel to normals of K.

Received December 10, 1958.

Since the case where the boundary has several components requires merely more verbiage but no new ideas, we consider, besides certain closed surfaces, only surfaces K homeomorphic to E^{n-1} whose boundaries B are homeomorphic to S^{n-2}.

If K is of class C^2 and has positive curvature, then its principal radii of curvature $R_1, \cdots, R_{n-1}$ are defined and finite. $\{R_1 \cdots R_m\}$ $(1 \leq m \leq n-1)$ denotes the elementary symmetric function $R_1 R_2 \cdots R_m + \cdots$ of the R_i of which $R_1 \cdots R_m$ is one summand. Our first goal is a general form of the following statement:

I'. *Let* K, K' *be two convex hypersurfaces of class* C^2 *with positive Gauss curvature, with the same spherical image and the same boundary* B *of class* C^1. *If for some fixed* m *the function* $\{R_1 \cdots R_m\}$ *has the same value at points of* K *and* K' *with parallel normals, then* K = K'.

Consider a closed convex hypersurface K^*, and on K^* a set B homeomorphic to S^{n-2}. Let B decompose K^* into the sets K and C^0 homeomorphic to E^{n-1}, put $C = C^0 \cup B$, and assume that the spherical images X_K and X_C of K and C are disjoint.

We follow the method of [5], found also in [3, Sections 8, 9]. For each m $(1 \leq m \leq n-1)$, an area function $a_m(K^*, X)$ is defined for all Borel sets X on Σ. We agree to *consider only Borel sets*. For K^* of class C^2 with positive curvature,

$$a_m(K^*, X) = \int_X \{R_1 \cdots R_m\}\, d\Sigma_u .$$

For $X \subset \Sigma$, we define

$$a_m(K, X) = a_m(K^*, X \cap X_K), \qquad a_m(C, X) = a_m(K^*, X \cap X_C).$$

Because of $X_K \cap X_C = 0$ and $X_K \cup X_C = \Sigma$, we have

$$a_m(K, X) + a_m(C, X) = a_m(K^*, X). \tag{1}$$

From the definition of $a_m(K^*, X)$ in [5] or [3], we deduce that the functions $a_m(K, X)$ and $a_m(C, X)$ are independent of K^* in this sense: If K lies on a second convex surface $K_1^* = K \cup C_1$ with $X_K \cap X_{C_1} = 0$, then $a_m(K, X)$ is the same function evaluated for K_1^* as for K^*. The same is true for $a_m(C, X)$ if $K_1^* = K_1 \cup C$, $X_{K_1} \cap X_C = 0$ and $X_K = X_{K_1}$, because then X_C is the same for K^* and K_1^*.

In particular, if no supporting plane of K contains a point of B, then the spherical images of K and C will be disjoint, no matter in which surface we imbed K. In that case, $a_m(K, X)$ depends therefore only on K, and $a_m(C, X)$ will, with the above notation, be the same function for K and any other K_1 which does not have supporting planes containing points of B, provided $X_K = X_{K_1}$.

We call $a_m(K, X)$ *the* m-*th area function of* K. For K of class C^2 and with positive curvature, we have

$$a_m(K, X) = \int_{X \cap X_K} \{R_1 \cdots R_m\}\, d\Sigma_u , \tag{2}$$

and for general K we see from [5] that $a_m(K, X)$ can be obtained by approximation of K^* with smooth convex surfaces as a limit of integrals (2).

3. UNIQUENESS FOR SURFACES WITH A GIVEN BOUNDARY

With the help of $a_m(K, X)$ we can now formulate our generalization of I':

THEOREM I. *Let* K, K' *be convex hypersurfaces in* E^n, *homeomorphic to* E^{n-1} *and with the same boundary* B *homeomorphic to* S^{n-2}, *such that the supporting planes of* K *and* K' *do not contain points of* B. *If* K *and* K' *have the same spherical image and the same* m-*th area function* (m *fixed*), *then* K = K'.

The assumption on the area function means that

$$a_m(K, X) = a_m(K', X) \qquad \text{for every } X \subset \Sigma.$$

If B lies in a hyperplane H, then it is a closed convex surface in H. If K coincides with the convex domain J bounded by B in H, then its spherical image consists of one point. Hence that of K' consists of the same point, so that K' lies in H and also coincides with J. For $B \subset H$ but $K \neq J$, the surfaces K, K' lie on the same side of H because they have the same spherical image. We denote by D and D' the closed convex surfaces obtained from K and K' by adding B and J.

We define D, D' similarly when B does not lie in a hyperplane, namely as the boundaries of the convex closures of $K \cup B$ and $K' \cup B$. Then $K \cup B \subset D$, because any subset of a convex surface lies on the boundary of its own convex closure. Put $C^0 = D - (K \cup B)$. A supporting plane H_p of D at a point p of C^0 is a supporting plane of B. For $H_p \cap (K \cup B)$ contains a point q (see [2, p. 6]), and the segment from p to q lies on H_p and contains a point of B; actually $q \in B$ because of our hypothesis on the supporting planes of K.

Thus C^0 is one of the two sets C^0, C_1^0 homeomorphic to E^{n-1} into which B decomposes the boundary of its own convex closure. By the same argument, D' is either $K' \cup B \cup C^0$ or $K' \cup B \cup C_1^0$. Since K and K' have the same spherical image, $D' = K' \cup B \cup C^0$.

By (1) and the ensuing discussion, the surfaces D and D' have the same m-th area function

$$a_m(D, X) = a_m(K, X) + a_m(C, X) = a_m(K', X) + a_m(C, X) = a_m(D', X).$$

Since D and D' do not lie in hyperplanes, we conclude from the results of [1] or [5] (see also [3, p. 70]) that D' originates from D by a translation. But D and D' have $C^0 \cup B$ in common; hence K = K'.

4. UNIQUENESS OF CAPS

The applicability of the theory of convex bodies to Theorem I rested essentially on the fact that this theory is not restricted to smooth surfaces. We give a second application of this fact.

A *convex cap* is a convex surface K homeomorphic to E^{n-1} whose boundary B is homeomorphic to S^{n-2} and lies in a hyperplane H, and whose normal projection on H falls on the domain J bounded by B, or on B. The cap is *spatial* when different from J, and *proper* when its projection is J. Certain, but not all, proper caps satisfy the hypothesis of Theorem I, so that their m-th area functions $a_m(K, X)$ are defined. Whether $a_m(K, X)$ determines a spatial K among all caps up to translations is not known and seems doubtful except for $m = n - 1$, but we shall define new functions $a_m^0(K, x)$, with $a_{n-1}^0(K, X) = a_{n-1}(K, X)$ for proper K, which determine K.

The restriction to spatial caps is essential, since caps in parallel hyperplanes and with the same area have the same $(n - 1)$-st area function. We consider improper caps because we do not only aim at a uniqueness theorem, but also at an existence theorem for given a_{n-1}^0, and there does not seem to exist a simple way of distinguishing general proper caps from certain improper ones in terms of a_{n-1}^0.

With the previous notations, let K be a spatial cap and K_1 its image under reflection in H. Then $K_* = K \cup B \cup K_1$ is a closed convex surface. Denote the unit normal to H towards the side of K by v, and by Σ_v^+, Σ_v^-, G_v respectively the subsets $x \cdot v > 0$, $x \cdot v < 0$, $x \cdot v = 0$ of Σ. We define

$$a_m^0(K, X) = a_m(K_*, \Sigma_v^+ \cap X) + a_m(K_*, G_v \cap X)/2 . \tag{3}$$

Then

$$a_m^0(K_1, X) = a_m(K_*, \Sigma_v^- \cap X) + a_m(K_*, G_v \cap X)/2$$

and

$$a_m^0(K, X) + a_m^0(K_1, X) = a_m(K_*, X) . \tag{4}$$

In order to discuss the meaning of a_{n-1}^0, denote by Z the cylinder formed by the lines normal to H at the points of B. Then (see [5])

$$a_{n-1}(K_*, G_v) = \text{area } Z \cap K_* . \tag{5}$$

For proper caps, $Z \cap D = B$, hence $a_{n-1}(K_*, G_v) = 0$.

Also, $\Sigma_v^+ \supset X_K$ for proper K, hence

$$a_m(K, X) = a_m(K_*, X_K \cap X) \leq a_m(K_*, \Sigma_v^+ \cap X) .$$

But

$$a_{n-1}(K_*, \Sigma_v^+ - X_K) = 0 ,$$

because this number represents the Minkowski area (see [5]) of the points of K_* with normals in $\Sigma_v^+ - X_K$, and all these points lie in B. Thus

$$a_{n-1}^0(K, X) = a_{n-1}(K, X) \qquad \text{for proper caps.}$$

The definition (3) of $a_{n-1}^0(K, X)$ as area function for improper caps is also entirely natural in view of (5), but (3) involves for $m < n - 1$ the behavior of K at the boundary in a non-obvious way. Our uniqueness theorem for caps is:

THEOREM II. *If two spatial caps* K, K' *have the same* m*-th area function* ($a_m^0(K, X) = a_m^0(K', X)$ *for all* $X \subset \Sigma$), *then* K' *originates from* K *by a translation.*

We define K_1', K_*' for K' in the same way as K_* was defined for K. Then, for any set $X \subset \Sigma$ and its image X_1 under reflection in $x \cdot v = 0$,

$$a_m(K_*, X) = a_m(K_*, X_1), \qquad a_m(K'_*, X) = a_m(K'_*, X_1),$$

therefore also

$$a^0_m(K_1, X) = a_m(K'_1, X),$$

and by (4),

$$a_m(K_*, X) = a_m(K'_*, X) \qquad \text{for all } X \subset \Sigma;$$

hence it follows from the uniqueness theorem in [1] and [5] (see also [3, p. 70]), that K'_* originates from K_* by a translation. The planes H and H' containing the boundaries of K and K' and also the sides of H, H' containing K and K' correspond under the translation, because the images of Σ^+_v on K_* and K'_* correspond. Moreover, $a^0_m(K, X) > 0$ for some $X \subset \Sigma^+_v$, $a^0_m(K, X) = 0$ for all $X \subset \Sigma^-_v$, and similarly for K', which defines the sides of H and H' on which K and K' lie.

As an application we observe that for $n = 3$ this theorem and the intrinsicness of the extrinsic curvature (see [3, p. 107]) imply that a spherical cap is, up to motions, determined by its intrinsic metric. This is a special case of a general theorem of Pogorelov (for references see [3, p. 166]). The same holds for spherical caps in E^n for odd $n > 3$ within the class of those smooth caps for which the intrinsicness of the Gauss-curvature $R_1^{-1} \cdots R_{n-1}^{-1}$ has been shown.

5. EXISTENCE OF CAPS WITH GIVEN $a^0_{n-1}(K, X)$

Any closed convex surface K_* satisfies

$$\int_\Sigma u_i \, a_m(K_*, d\Sigma_u) = 0 \qquad (i = 1, \cdots, n). \tag{6}$$

For $m < n - 1$, it is known that the condition

$$\int_\Sigma u_i \, a(d\Sigma_u) = 0$$

is not sufficient for a non-negative completely additive set function $a(X)$ on Σ to be the m-th area function of a convex surface; but necessary and sufficient conditions have not been found (see [3, Section 9]). For a^0_{n-1} we have the following result:

THEOREM III. *Let* $a(X)$ *be a non-negative completely additive set function defined on all Borel sets* X *on* Σ *satisfying the conditions*

(a) $$a(X) = 0, \textit{ when } X \textit{ lies in } x_n < 0,$$

(b) $$\int_\Sigma u_j \, a(d\Sigma_u) = 0 \textit{ for } j < n,$$

(c) $$a(G_u) < a(\Sigma) \textit{ for all } u$$

(*where* G_u *denotes, of course, the intersection of* Σ *with* $u \cdot x = 0$).

Then there exists, up to translations, exactly one spatial convex cap K *with* $a^0_{n-1}(K, X) = a(X)$ *for all Borel sets* $X \subset \Sigma$, *and its boundary lies in a plane* $x_n = \text{const}$.

The condition (c) is necessary to prevent degeneracy (compare [3, p. 63]). Denote by X_1 the image of the set $X \subset \Sigma$ under reflection in the hyperplane $H: x_n = 0$. We define a new set function $F(X)$ by

$$F(X) = a(X) + a(X_1),$$

then $F(X) = F(X_1)$, and $F(X)$ satisfies the conditions

$$\int_\Sigma u_i F(d\Sigma_u) = 0 \qquad (i = 1, \cdots, n).$$

If $u^* = (u_1, \cdots, u_{n-1}, -u_n)$, then for any u

$$F(G_u) = a(G_u) + a(G_{u^*}) < 2a(\Sigma) = F(\Sigma);$$

hence there exists a closed convex surface K_* with

$$a_{n-1}(K_*, X) = F(X) \qquad \text{for all } X \subset \Sigma$$

(see [1], [5] and [3, p. 64]).

We claim that K_* is symmetric with respect to a plane H' parallel to H. For if K_{*1} is the image of K_* under reflection in H, then

$$a_{n-1}(K_*, X_1) = a_{n-1}(K_*, X) = a_{n-1}(K_{*1}, X_1),$$

so that K_{*1} originates from K_* by a translation, whence the assertion follows.

Thus H' decomposes K_* into two spatial caps K, K_1 symmetric to H'. By construction, when $v = (0, \cdots, 0, 1)$ is a normal of K,

$$a^0_{n-1}(K, X) = a(X) \qquad \text{for } X \subset \Sigma^+_v,$$

$$a^0_{n-1}(K, X) = 0 = a(X) \qquad \text{for } X \subset \Sigma^-_v,$$

and also

$$a^0_{n-1}(K, X) = F(X)/2 = a(X) \qquad \text{for } X \subset G_v,$$

because $F(X)$ has for $X \subset G_v$ the following geometric meaning: consider the set Y of those points in $B = K_* \cap H'$ where the normal to B in H' falls in X, and let Z be the cylindrical set consisting of the lines normal to H at points of Y. Then $F(X)$ is the area of the set $Z \cap K_*$.

The last three equations and the additivity of $a^0_{n-1}(K, X)$ prove that

$$a^0_{n-1}(K, X) = a(X) \qquad \text{for } X \subset \Sigma.$$

The uniqueness, up to translations, is contained in Theorem II.

The question arises whether the cap K obtained in this way is smooth when the data are smooth. Nothing is known for $n > 3$. For $n = 3$ we can say the following: For any Borel subset X of the connected open subset Y of Σ_v^+, let

$$a_2^0(K, X) = \int_X f(u)\, d\Sigma_u,$$

where f(u) is defined on Y and is positive and continuous. Then there is exactly one point $x \in K$ with normal u, so that x may be considered as function x(u) on Y. Moreover, $f^{-1}(u)$ is the Gauss curvature of K at x if defined as the limit of the ratio of the area of a set on Σ shrinking to u and the area of the corresponding set on K. (This does *not* imply that the part of K corresponding to Y is of class C^2, see [3, p. 29]).

A theorem of Pogorelov [9], with an improvement due to Nirenberg (see [3, p. 36]) implies: *If* $f(u) = f(u_1, u_2, (1 - u_1^2 - u_2^2)^{1/2}$ *is on* Y *positive and of class* C^m $(m \geq 2)$, *then the components* $x_i(u_1, u_2, (1 - u_1^2 - u_2^2)^{1/2})$ *of* x(u) *are* (all as functions of u_1, u_2) *at least of class* C^{m+1} (*analytic with* f). The following special case merits being formulated explicitly:

If $f(u_1, u_2, u_3)$ $(\Sigma u_i^2 = 1)$ is positive and continuous for $u_3 > 0$, vanishes for $u_3 \leq 0$, and satisfies

$$\int_\Sigma u_i f(u)\, d\Sigma_u = 0 \qquad (i = 1, 2).$$

then there is, up to translations, only one proper cap K with

$$a_2(K, X) = a_2^0(K, X) = \int_X f(u)\, d\Sigma_u.$$

The boundary of K lies in a plane parallel to $x_3 = 0$, and $f^{-1}(u)$ is the Gauss curvature of K defined as area limit.

The cap is of class C^{m+1} $(m \geq 2)$, or analytic, if $f(u_1, u_2, (1 - u_1^2 - u_2^2)^{1/2})$ is of class C^m or analytic.

All assertions have been proved before, except that K is proper when f(u) is merely positive and continuous. This follows from a theorem of Alexandrov (see [3, p. 34, Theorem 5.4]).

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