ON THE SEMI-SIMPLECTICITY OF GROUP ALGEBRAS

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Let $F$ be a field of characteristic zero, and let $G$ be an arbitrary group. For the real or complex field, the semi-simplicity of the discrete group algebra $F[G]$ has been established by means of analytical results on Banach algebras ("semi-simplicity" will always be used in the sense of Jacobson, and "radical" will mean "Jacobson radical"; for definitions and for references concerning this problem, see [3, especially Chapter I, p. 22, Problem 1]). The general case of a field $F$ of characteristic zero which is of infinite transcendence degree has been disposed of in [1] by reduction to the complex case. Commutative groups $G$ and groups of a slightly more general structure have been treated by Villamayor in [5], with the help of cohomological methods. In the present note we obtain an algebraic proof for all known cases, together with a slight extension.

1. First we observe the following simple fact.

**Lemma 1.** Let $\mathbb{Q}$ be the field of all rational numbers. Then $\mathbb{Q}[G]$ does not contain nonzero nil ideals.

Indeed, let $0 \neq x = \sum x_g g \in \mathbb{Q}[G]$, and let $x^* = \sum x_g g^{-1}$; then $y = xx^* = \sum y_g g \neq 0$, with $y_g^{-1} = y_g^{-1}$ and $y_g = \sum x_g^2 \neq 0$. One readily verifies that if $z = \sum z_g g$ has the property that $z_g \neq 0$ and $z_g = z_g^{-1}$, then $z^2$ has the same property. Now, if $x \neq 0$ belongs to a nil ideal, then $y = xx^*$ is also nil, but in view of the preceding remark, $y^{2n}$ is never zero. This is a contradiction.

In view of Theorems I and II of [1], Lemma 1 yields the following proposition.

**Theorem 1.** If $F$ is a transcendental extension of $\mathbb{Q}$, then $F[G]$ is semi-simple.

For let $F$ contain $P$, a pure transcendental extension of $\mathbb{Q}$ such that $F$ over $P$ is algebraic. Then it follows by [1, Theorem II] that the radical of $P[G]$ is $N \otimes P$, where $N$ is a nil ideal of $\mathbb{Q}[G]$, and the previous lemma shows that $N = 0$. Consequently, $P[G]$ is semi-simple. Now $F$ is a separable algebraic extension of $P$, hence, by [1, Theorem I], $F[G]$ is semi-simple.

This extends the result of [1], where $F$ was assumed to be of infinite transcendence degree over $\mathbb{Q}$ and where a reduction to the complex case was used.

2. Consider the case where $H$ is a finitely generated group. Here $\mathbb{Q}[H]$ is generated by the generators of $H$ and their inverses; that is, $\mathbb{Q}[H]$ is a finitely generated algebra. In particular, it is well known that the radical of a finitely generated commutative ring is nil. (A generalization of this result is proved in [2, Theorem 5]. The commutative case is equivalent to the Hilbert Nullstellensatz; for reference see [3, p. 23, Problem 4].) Hence, in view of Lemma 1, there follows

**Lemma 2.** If $H$ is a finitely generated commutative group, then $\mathbb{Q}[H]$ is semi-simple.

The case of an arbitrary group can be reduced to finitely generated groups.

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THEOREM 2. If $F[H]$ is semi-simple, for each finitely generated subgroup of $G$, then $F[G]$ is semi-simple.

This follows from the fact that if an element $x = \Sigma x_g g \in F[G]$ has an inverse in $F[G]$, then it has also an inverse in $F[H]$, where $H$ denotes the subgroup of $G$ generated by the finite number of $g$'s for which $x_g \neq 0$.

For if $xy = e$ and $y = \Sigma y_g g$, then

(*)

$$\Sigma x_g y_{g^{-1}k} = 1, \quad \Sigma x_g y_g^{-1k} = 0$$

for all $k \in G$. Let $y' = \Sigma y_h h$, where the sum ranges only over all $h \in H$. Then

$$xy' = \Sigma (\Sigma x_g y_{g^{-1}k}) k = e.$$  

Indeed, if $k \notin H$, then none of $g^{-1}k \in H$ for all $g$ with $x_g \neq 0$, and therefore the corresponding coefficient is zero; and if $k \in H$, then $g^{-1}k \in H$ for all of these $g$'s, and our result follows by (*). The proof is complete, since $y' \in F[H]$. 

Now, if $G$ satisfies the requirement of the lemma and $x = \Sigma x_g g$ belongs to the radical of $F[G]$, then the preceding remark clearly shows that the inverse of $e - xz$ ($z \in F[H]$) also belongs to $F[H]$; but $F[H]$ is semi-simple, hence $x = 0$. Thus $F[G]$ is also semi-simple.

From Lemma 2 and Theorem 2 we obtain the following proposition.

COROLLARY 1. If $G$ is a commutative group, then $Q[G]$ is semi-simple.

Now [1, Theorem V] enables us to settle the problem for arbitrary fields of characteristic zero:

THEOREM 3. If $F$ is a field of characteristic zero and $G$ is commutative, then $F[G]$ is semi-simple.

3. Remarks. 1) There is a conjecture that the Jacobson radical of a finitely generated ring is always nil. The confirmation of this conjecture would yield, by the methods of the previous section, that $F[G]$ is semi-simple for an arbitrary field $F$ of characteristic zero and an arbitrary group $G$.

2) In [2] it was shown that the Jacobson radical of a finitely generated ring which satisfies an identity is nil. Now the group ring $F[H]$ satisfies an identity if all primitive representations of $H$ are of bounded degree. (For a definition and the discussion of such groups and representations, see I, Kaplansky [4].) Thus, following the proof in the previous section, one can clearly obtain

COROLLARY 2. If every finitely generated subgroup of $G$ has the property that all its primitive representations are of bounded degree, and if $F$ is a field of characteristic zero, then $F[G]$ is semi-simple.

If $G$ is a group with a center $Z$ such that $G/Z$ is locally finite, then one readily verifies that $G$ satisfies Corollary 2.

3 The following remark is due to Villamayor: The semi-simplicity of $F[G]$ for an arbitrary field of characteristic zero yields the semi-simplicity of $K[G]$ for an arbitrary semi-simple commutative algebra $K$ over the rational numbers. For such algebras are subdirect sums of fields $F$ of characteristics zero, and therefore $K[G]$ is also a subdirect sum of semi-simple rings $F[G]$. Consequently $K[G]$ is semi-simple.
REFERENCES


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