ON THE SEMI-SIMPLICITY OF GROUP ALGEBRAS

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Let F be a field of characteristic zero, and let G be an arbitrary group. For the real or complex field, the semi-simplicity of the discrete group algebra F[G] has been established by means of analytical results on Banach algebras ("semi-simplicity" will always be used in the sense of Jacobson, and "radical" will mean "Jacobson radical"; for definitions and for references concerning this problem, see [3, especially Chapter I, p. 22, Problem 1]). The general case of a field F of characteristic zero which is of infinite transcendence degree has been disposed of in [1] by reduction to the complex case. Commutative groups G and groups of a slightly more general structure have been treated by Villamayor in [5], with the help of cohomological methods. In the present note we obtain an algebraic proof for all known cases, together with a slight extension.

1. First we observe the following simple fact.

LEMMA 1. *Let* Q *be the field of all rational numbers. Then* Q[G] *does not contain nonzero nil ideals.*

Indeed, let $0 \neq x = \Sigma x_g g \in Q[G]$, and let $x^* = \Sigma x_g g^{-1}$; then $y = xx^* = \Sigma y_g g \neq 0$, with $y_{g^{-1}} = y_g$ and $y_e = \Sigma x_g^2 \neq 0$. One readily verifies that if $z = \Sigma z_g g$ has the property that $z_e \neq 0$ and $z_g = z_{g^{-1}}$, then z^2 has the same property. Now, if $x \neq 0$ belongs to a nil ideal, then $y = xx^*$ is also nil, but in view of the preceding remark, y^{2^n} is never zero. This is a contradiction.

In view of Theorems I and II of [1], Lemma 1 yields the following proposition.

THEOREM 1. *If* F *is a transcendental extension of* Q, *then* F[G] *is semi-simple.*

For let F contain P, a pure transcendental extension of Q such that F over P is algebraic. Then it follows by [1, Theorem II] that the radical of P[G] is $N \otimes P$, where N is a nil ideal of Q[G], and the previous lemma shows that $N = 0$. Consequently, P[G] is semi-simple. Now F is a separable algebraic extension of P, hence, by [1, Theorem I], F[G] is semi-simple.

This extends the result of [1], where F was assumed to be of infinite transcendence degree over Q and where a reduction to the complex case was used.

2. Consider the case where H is a finitely generated group. Here Q[H] is generated by the generators of H and their inverses; that is, Q[H] is a finitely generated algebra. In particular, it is well known that the radical of a finitely generated commutative ring is nil. (A generalization of this result is proved in [2, Theorem 5]. The commutative case is equivalent to the Hilbert Nullstellensatz; for reference see [3, p. 23, Problem 4].) Hence, in view of Lemma 1, there follows

LEMMA 2. *If* H *is a finitely generated commutative group, then* Q[H] *is semi-simple.*

The case of an arbitrary group can be reduced to finitely generated groups:

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THEOREM 2. *If* F[H] *is semi-simple, for each finitely generated subgroup of* G, *then* F[G] *is semi-simple.*

This follows from the fact that if an element $x = \Sigma x_g g \in F[G]$ has an inverse in F[G], then it has also an inverse in F[H], where H denotes the subgroup of G generated by the finite number of g's for which $x_g \neq 0$.

For if $xy = e$ and $y = \Sigma y_g g$, then

$$(*) \qquad \sum x_g y_{g^{-1}} = 1, \qquad \sum x_g y_{g^{-1}k} = 0$$

for all $k \in G$. Let $y' = \Sigma y_h h$, where the sum ranges only over all $h \in H$. Then

$$xy' = \sum \left(\sum x_g y_{g^{-1}k}\right) k = e.$$

Indeed, if $k \notin H$, then none of $g^{-1}k \in H$ for all g with $x_g \neq 0$, and therefore the corresponding coefficient is zero; and if $k \in H$, then $g^{-1}k \in H$ for all of these g's, and our result follows by (*). The proof is complete, since $y' \in F[H]$.

Now, if G satisfies the requirement of the lemma and $x = \Sigma x_g g$ belongs to the radical of F[G], then the preceding remark clearly shows that the inverse of $e - xz$ ($z \in F[H]$) also belongs to F[H]; but F[H] is semi-simple, hence $x = 0$. Thus F[G] is also semi-simple.

From Lemma 2 and Theorem 2 we obtain the following proposition.

COROLLARY 1. *If* G *is a commutative group, then* Q[G] *is semi-simple.*

Now [1, Theorem V] enables us to settle the problem for arbitrary fields of characteristic zero:

THEOREM 3. *If* F *is a field of characteristic zero and* G *is commutative, then* F[G] *is semi-simple.*

3. *Remarks.* 1) There is a conjecture that the Jacobson radical of a finitely generated ring is always nil. The confirmation of this conjecture would yield, by the methods of the previous section, that F[G] is semi-simple for an arbitrary field F of characteristic zero and an arbitrary group G.

2) In [2] it was shown that the Jacobson radical of a finitely generated ring which satisfies an identity is nil. Now the group ring F[H] satisfies an identity if all primitive representations of H are of bounded degree. (For a definition and the discussion of such groups and representations, see I, Kaplansky [4].) Thus, following the proof in the previous section, one can clearly obtain

COROLLARY 2. *If every finitely generated subgroup of* G *has the property that all its primitive representations are of bounded degree, and if* F *is a field of characteristic zero, then* F[G] *is semi-simple.*

If G is a group with a center Z such that G/Z is locally finite, then one readily verifies that G satisfies Corollary 2.

3 The following remark is due to Villamayor: The semi-simplicity of F[G] for an arbitrary field of characteristic zero yields the semi-simplicity of K[G] for an arbitrary semi-simple commutative algebra K over the rational numbers. For such algebras are subdirect sums of fields F of characteristics zero, and therefore K[G] is also a subdirect sum of semi-simple rings F[G]. Consequently K[G] is semi-simple.

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