THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS ON HOROCYCLIC GROUPS, II

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1. INTRODUCTION

In a recent paper [3], I showed how the circle method can be used to determine the Fourier coefficients of entire automorphic forms on certain horocyclic groups (*Grenzkreisgruppen*). These groups are characterized by the fact that they are finitely generated and that they possess a single equivalence class of parabolic vertices, or, to put it in another way, that they have fundamental regions which touch the real axis in exactly one real point (possibly ∞). In the present paper, we extend the method to all finitely generated horocyclic groups.

More precisely, a horocyclic group Γ is a group of linear transformations of a complex variable such that

(a) Γ is discontinuous in the upper half-plane but is not discontinuous at any point of the real axis,

(b) every transformation of Γ preserves the upper half-plane.

If, in addition,

(c) Γ contains translations,

(d) Γ is finitely generated,

we call Γ an H-group. (In [3], Γ was defined to be an H-group if it satisfies conditions (a) to (c).) In [3], we restricted Γ by the further condition that all its parabolic vertices (fixed points of parabolic transformations of Γ) should be equivalent under Γ (and therefore, by (c), equivalent to ∞). Here we shall lift this restriction.

By an entire automorphic form of real dimension r on Γ, we mean an analytic function $F(z)$ of a complex variable z, which is regular in the upper half-plane, which satisfies there a transformation equation

$$(1.1) \qquad F(Vz) = \varepsilon(V)\, m(V, z)\, F(z), \qquad m(V, z) = (cz + d)^{-r}$$

for every $V \in \Gamma$, and which tends to a definite limit (finite or infinite) as z approaches a parabolic point from within a fundamental region. (Note that equation (1.1) is written slightly differently in [3].) The last condition is equivalent to the following: $F(z)$ has at most a pole (not an essential singularity) in the local variable appropriate to a given parabolic point.

In (1.1), the multiplier $\varepsilon(V)$ is of absolute value 1 and is independent of z. The branch of the many-valued function $m(V, z)$ is determined in the following way: If w is a complex number and s is real, we define

$$(1.2) \qquad w^s = |w|^s \exp(i\, s \arg w) \qquad (-\pi \leq \arg w < \pi).$$

For any two substitutions $V_1, V_2 \in \Gamma$, we can evaluate $F(V_1 V_2 z)$ in two ways; comparison then yields a "consistency condition":

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(1.3) $$\varepsilon(V_1V_2)\,m(V_1V_2, z) = \varepsilon(V_1)\varepsilon(V_2)\,m(V_1, V_2z)\,m(V_2, z).$$

By means of the transformation equation (1.1), we show in Section 3 that at each finite parabolic point p of Γ, there is a Fourier series for $F(z)$:

(1.4) $$F(z) = (-1/\lambda(z-p))^{-r}\, e(-\alpha/\lambda(z-p))\, f_p(t), \qquad t = e(-1/\lambda(z-p)),$$
$$f_p(t) = \sum_{m=-\mu}^{\infty} a_m^{(p)} t^m,$$

where λ, α, μ are constants associated with the form F and the point p (see (2.2), (3.2), (3.3)), and where

$$e(u) = e^{2\pi i u}.$$

In [3], we had only one inequivalent parabolic point in the fundamental region of Γ (namely, ∞) and thus only one expansion (1.4). We determined the a_m ($m \geq 0$) from the a_m ($m < 0$) (coefficients of the "principal part" of $f(t)$), by using the expansion (1.4) about ∞. In the present case, we shall have to consider a set of inequivalent vertices $p_1, p_2, \cdots, p_s$ of a fundamental region of Γ, which gives rise to a set of expansions of the form (1.4). Again, we shall determine the Fourier coefficients $a_m^{(k)}$, $m \geq 0$, from the $a_m^{(j)}$ ($m < 0$; $j = 1, 2, \cdots, s$), this time by utilizing the set of expansions (1.4) about the vertices $p_1, \cdots, p_s$.

The circle method starts out by expressing $a_m^{(k)}$ as a Cauchy integral:

$$a_m^{(k)} = \frac{1}{2\pi i}\int_C \frac{f_k(t)}{t^{m+1}}\,dt.$$

The principal problem is then to select a path C which will permit us to take advantage of the transformation equation (1.1). This problem is solved in Section 4. Once the path has been chosen, the work proceeds by straightforward methods.

We collect our main results in the theorems which follow. First, we need some definitions.

Let Γ be an H-group. Fix a fundamental region R_0 of Γ which does not contain ∞ as a vertex. Let $p_1, p_2, \cdots, p_s$ be a complete set of inequivalent parabolic vertices of R_0. For each j, the subgroup Γ_j of Γ which preserves p_j is generated by a matrix

$$S_j = \begin{pmatrix} 1+\lambda_j p_j & -\lambda_j p_j^2 \\ \lambda_j & 1-\lambda_j p_j \end{pmatrix}$$

(see (2.2)). We select a system of elements V of Γ such that

$$A_j V A_k^{-1} = \begin{pmatrix} a_{jk} & \\ c_{jk} & d_{jk} \end{pmatrix}$$

satisfies

$$0 \leq -d_{jk}/c_{jk} < \lambda_k, \qquad 0 \leq a_{jk}/c_{jk} < \lambda_j.$$

(A_j, A_k are defined in (2.1).) This set of V we call M_{jk}. The set of c_{jk} for which $V \in M_{jk}$ is denoted by C_{jk}, that is,

$$C_{jk} = \left\{ x \,\middle|\, \exists \begin{pmatrix} \cdot & \cdot \\ x & \cdot \end{pmatrix} \in A_j \Gamma A_k^{-1} \right\},$$

where, as usual, $A_j \Gamma A_k^{-1}$ denotes the set of elements X such that

$$X = A_j V A_k^{-1}, \qquad V \in \Gamma.$$

Also, let

$$\text{(1.5)} \qquad L(c, \sigma, \tau, r) = \begin{cases} (\tau/\sigma)^{(r+1)/2} I_{r+1}(4\pi\sigma^{1/2}\tau^{1/2}/c) & (\sigma > 0,\ r > 0), \\ (\tau/\sigma)^{(r+1)/2} I_{-r-1}(4\pi\sigma^{1/2}\tau^{1/2}/c) & (\sigma > 0,\ r < -2), \\ (2\pi\tau/c)^{r+1}/\Gamma(r+2) & (\sigma = 0), \end{cases}$$

so that $L(c, 0, \tau, r) = \lim_{\sigma \to 0} L(c, \sigma, \tau, r)$ for $r > 0$. Here $I_r(z)$ is the Bessel function

$$I_r(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{r+2n}}{\Gamma(r+n+1)\Gamma(n+1)}.$$

Moreover, we set

$$\text{(1.6)} \qquad m_k = (m + \alpha_k)/\lambda_k, \qquad \nu_j = (\nu - \alpha_j)/\lambda_j \qquad (1 \leq j, k \leq s)$$

(see (2.2), (3.2)), and for $V \in M_{jk}$,

$$\text{(1.7)} \qquad A(c_{jk}, \nu, m) = \sum_{d_{jk} \in D(c_{jk})} \varepsilon^{-1}(V)\, e\{(m_k d_{jk} - \nu_j a_{jk})/c_{jk}\}.$$

We are now ready to state our results.

THEOREM 1. *Let* F(z) *be an automorphic form of dimension* $r > 0$ *on* Γ *and having the expansions*

$$F(z) = (-1/\lambda_k(z - p_k))^{-r} t_k^{\alpha_k} f_k(t_k), \qquad t_k = e(-1/\lambda_k(z - p_k)),$$

$$\text{(1.8)} \qquad f_k(t) = \sum_{m=-\mu_k}^{\infty} a_m^{(k)} t^m \qquad (1 \leq k \leq s)$$

about the parabolic vertices $p_1, p_2, \cdots, p_s$. *Then, for each* k $(1 \leq k \leq s)$, *the Fourier coefficients* $a_m^{(k)}$ *with* $m \geq 0$ *are given in terms of the set of coefficients* $a_m^{(j)}$ *with* $m < 0$ $(1 \leq j \leq s)$ *by the formula*

$$(1.9)\quad a_m^{(k)} = \left(\frac{2\pi}{\lambda_k}\right) e(r/4) \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{\substack{c_{jk}\in C_{jk}\\ c_{jk}>0}} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k) L(c_{jk}, m_k, \nu_j, r).$$

Theorem 1 was obtained by Petersson [4], who used his generalized Poincaré series and the Hilbert space of automorphic forms. Theorems 2 and 3 are new, so far as I am aware.

THEOREM 2. *If* F(z) *is an automorphic form of dimension zero on* Γ *having the expansions* (1.8), *then, for* $m \geq 1$,

$$(1.10)\quad a_m^{(k)} = \left(\frac{2\pi}{\lambda_k}\right) e(r/4) \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{\substack{c_{jk}\in C_{jk}\\ 0<c_{jk}<\beta\sqrt{m}}} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k) L(c_{jk}, m_k, \nu_j, r) + O(1),$$

where β *is any positive constant.*

THEOREM 3. *Let* G(z) *be an automorphic form of dimension* -2 *on* Γ *with Fourier coefficients* $b_m^{(k)}$ *which is, moreover, the derivative of a form* F(z) *(of dimension zero). Then, for* $m \geq 1$,

$$(1.11)\quad b_m^{(k)} = \left(\frac{2\pi}{\lambda_k}\right)^2 i\, e(r/4)(m+\alpha_k) \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{\substack{c_{jk}\in C_{jk}\\ 0<c_{jk}<\beta\sqrt{m}}} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k) \times L(c_{jk}, m_k, \nu_j, r) + O(m).$$

Theorem 3 is obtained from Theorem 2 by merely differentiating (1.4).

The next two theorems are easy corollaries of Theorems 1 and 2.

THEOREM 4. *The Fourier coefficients of an automorphic form of nonnegative dimension on* Γ *satisfy the inequality*

$$(1.12)\quad a_m^{(k)} = O(m^{-r/2-3/4} \exp 4\pi\kappa (m+1)^{1/2}) \quad (m \to \infty),$$

where (see (2.5))

$$\kappa = \max_{1\leq j\leq s} \frac{((\mu_j - \alpha_j)\ \lambda_j \lambda_k)^{1/2}}{\overline{c_{jk}}}.$$

THEOREM 5. *An automorphic form of positive dimension on* Γ *which is finite at all parabolic vertices is identically zero.*

Theorem 4 is proved by applying the standard asymptotic formula for the Bessel function to (1.9) and (1.10). In Theorem 5, we are given that $a_m^{(j)} = 0$ for $m < 0$ and $j = 1, 2, \cdots, s$. Then (1.9) shows that $a_m^{(k)} = 0$ for $m \geq 0$.

These results generalize those of Zuckerman [5], who studied H-groups which are subgroups of the modular group. They are approximately co-extensive with the theorems of Petersson [4] insofar as entire automorphic forms are concerned. I shall treat meromorphic forms by the circle method in a later publication.

Finally, we can discuss forms of certain negative dimensions as we did in [3]. Let $F(z)$ be an automorphic form of dimension $r < -2$ on Γ having the expansions (1.4), where the coefficients $a_m^{(k)}$ $(m < 0,\ 1 \le k \le s)$ of the principal parts are given, as well as the α_k (see (3.2)). Assume all $\alpha_k > 0$. Construct the functions

$$G_k(z) = (-1/\lambda_k(z - p_k))^{-r} \sum_{m=-\mu_k}^{\infty} a_m^{(k)} e(-(m + \alpha_k)/\lambda_k(z - p_k)) \qquad (1 \le k \le s),$$

with the $a_m^{(k)}$ of (1.9). When $r < -2$, the order of magnitude of the $a_m^{(k)}$ can be established as in Section 7. Then, exactly as in [3], we can prove that $G_k(z)$ is an automorphic form of dimension r on Γ. Thus $F(z) - G_k(z)$ is a form on Γ which vanishes at the cusp p_k. Hence, we have the following theorem.

THEOREM 6. *If* $F(z)$ *is an automorphic form of dimension* $r < -2$ *having positive* α_k $(1 \le k \le s)$, *then the Fourier coefficients of* $F(z)$ *in its expansion* (1.4) *about the parabolic vertex* p_k *differ from the* $a_m^{(k)}$ *of* (1.9) *by the coefficients of an automorphic form on* Γ (*of dimension* r) *which vanishes at* p_k.

2. THE GROUP Γ

Let Γ be an H-group. The elements of Γ will be represented by unimodular matrices $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries. We assume that $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$; hence, $-V = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in \Gamma$ whenever $V \in \Gamma$. We identify $\pm V$ with the linear transformation $Vz = (az + b)/(cz + d)$ and use the symbol Γ to denote either the group of linear transformations or the group of matrices. As a discontinuous group, Γ is *discrete*, that is, there is no sequence of different matrices in Γ which tends to the identity matrix.

A parabolic vertex (or parabolic point) of Γ is, by definition, the fixed point of a parabolic element of Γ. Let p be a finite parabolic vertex, Γ_p the subgroup of Γ which preserves p, and A_p the matrix

$$A_p = \begin{pmatrix} 0 & -1 \\ 1 & -p \end{pmatrix}, \tag{2.1}$$

where $p \ne \infty$, $A_\infty = I$. Thus, $A_p(p) = \infty$. There exists a parabolic transformation $z' = Sz$ of Γ which has p as a fixed point. Every such transformation is of the form ([2], p. 22)

$$\frac{1}{z' - p} = \frac{1}{z - p} + \lambda,$$

or

$$S = S(\lambda) = \begin{pmatrix} 1 + \lambda p & -\lambda p^2 \\ \lambda & 1 - \lambda p \end{pmatrix}.$$

That is, $S(\lambda) \in \Gamma_p$ and obviously so does $S(\lambda)^m = S(m\lambda)$ for every integer m. Since Γ_p, as a subgroup of Γ, is discrete, it follows that there is a positive value λ_p of λ such that $S(\lambda_p) \equiv S_p$ generates Γ_p:

$$\Gamma_p = \{S_p\}, \qquad S_p = \begin{pmatrix} 1+\lambda_p p & -\lambda_p p^2 \\ \lambda_p & 1-\lambda_p p \end{pmatrix}. \tag{2.2}$$

We shall denote the parabolic vertices of Γ by $p_1, p_2, \cdots$, and correspondingly, A_{p_1} by A_1, λ_{p_1} by λ_1, and so forth. Let p_j, p_k be two finite parabolic points of Γ. In later sections, we shall need to consider the set $A_j\Gamma A_k^{-1}$. Let C_{jk} be the set of third coefficients in the elements of $A_j\Gamma A_k^{-1}$, that is,

$$C_{jk} = \left\{ x \mid \exists \begin{pmatrix} \cdot & \cdot \\ x & \cdot \end{pmatrix} \in A_j\Gamma A_k^{-1} \right\}. \tag{2.3}$$

LEMMA 1. *The set* C_{jk} *is discrete* ($j, k = 1, 2, \cdots$).

Following Petersson, we introduce the symbolic notation

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad U^{\kappa} = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} \quad (\kappa \text{ real}).$$

Then

$$A_j S_j A_j^{-1} = U^{-\lambda_j}, \qquad A_k S_k A_k^{-1} = U^{-\lambda_k}. \tag{2.4}$$

If the lemma is false, there exists a sequence of different c_n in

$$X_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in A_j\Gamma A_k^{-1}$$

with $c_n \to c$ (c finite) as $n \to \infty$. We may assume $c_n \geq 0$. Now $X_n = A_j V_n A_k^{-1}$ with $V_n \in \Gamma$, and therefore $Y_n = A_j S_j^{q_n} V_n S_k^{r_n} A_k^{-1}$ also belongs to $A_j\Gamma A_k^{-1}$ (q_n, r_n integers). But using (2.4), we calculate

$$Y_n = U^{-q_n\lambda_j} X_n U^{-r_n\lambda_k} = \begin{pmatrix} a_n - q_n\lambda_j c_n & \\ c_n & d_n - r_n\lambda_k c_n \end{pmatrix} = \begin{pmatrix} a_n' & b_n' \\ c_n & d_n' \end{pmatrix}.$$

For each n, choose q_n, r_n so that

$$1 \leq a_n' < 1 + \lambda_j c_n, \qquad 1 \leq d_n' < 1 + \lambda_k c_n.$$

Since $c_n \to c$, it follows that the sequences $\{a_n\}$, $\{d_n\}$ are bounded, so that on a certain subsequence we have

$$c_m \to c, \qquad a_m' \to a, \qquad d_m' \to d$$

with $1 \leq a \leq 1 + \lambda_j c$ and $1 \leq d \leq 1 + \lambda_k c$.

Now if $c \neq 0$, we have immediately

$$\lim_{m\to\infty} b'_m = \frac{ad - 1}{c}.$$

In the case $c = 0$, we have, for large m,

$$0 \le b'_m < \frac{(1 + \lambda_j c_m)(1 + \lambda_k c_m) - 1}{c_m} = \lambda_j + \lambda_k + \lambda_j \lambda_k c_m < 2(\lambda_j + \lambda_k).$$

Thus $\{b'_m\}$ is bounded, and on a further subsequence, $b'_p \to b$.

In any case, then, $Y_p \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the matrix being unimodular. It follows that $U_p = Y_p Y_{p+1}^{-1} \to I$. If only a finite number of the U_p were different, we should have $U_p = I$ $(p > N)$ or $Y_p = Y_{p+1}$ $(p > N)$. This contradicts the assumption that the c_p are all distinct.

Hence, we have a sequence $\{Y_p\} = \{A_j V_p A_k^{-1}\}$ $(V_p \in \Gamma)$ such that $Y_p Y_{p+1}^{-1} \to I$. But $Y_p Y_{p+1}^{-1} = A_j V_p V_{p+1}^{-1} A_j^{-1}$. It follows that $W_p = V_p V_{p+1}^{-1} \to I$. Now if the sequence $\{W_p\}$ contained only a finite number of distinct elements, we would have $W_p = I$ $(p > N)$ or $V_p = V_{p+1}$ $(p > N)$. But this would imply $Y_p = Y_{p+1}$ $(p > N)$, which we have just seen is false. Therefore, $\{W_p\}$ contains a sequence of distinct elements of Γ which tends to the identity, and so Γ is not discrete. This contradiction completes the proof.

The lemma implies that the set of positive values of c in C_{jk} has a positive minimum. We call this minimum

$$\overline{c_{jk}} = \min\{c \mid c \in C_{jk},\ c > 0\}. \tag{2.5}$$

For later use, we shall need the set

$$D(c_{jk}) = \left\{ d_{jk} \,\middle|\, \begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_{jk} \end{pmatrix} \in A_j \Gamma A_k^{-1},\ 0 \le -d_{jk}/c_{jk} < \lambda_k \right\}, \tag{2.6}$$

where $c_{jk} \in C_{jk}$, $c_{jk} > 0$.

LEMMA 2. *For each positive* $c_{jk} \in C_{jk}$, *the set* $D(c_{jk})$ *is finite.*

Let $D^*(c_{jk})$ be the set of d_{jk} such that $\begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_{jk} \end{pmatrix} \in A_j \Gamma A_k^{-1}$. The lemma will follow if we show that $D^*(c_{jk})$ is discrete. If it is not, there is a sequence of different $d_n \to d \ne \infty$ for which $X_n = \begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_n \end{pmatrix} \in A_j \Gamma A_k^{-1}$. Then we can prove in the same way as before that Γ is not discrete.

3. AUTOMORPHIC FORMS ON Γ

Let $F(z)$ be an automorphic form of the type described in Section 1. At each finite parabolic vertex p_k of Γ, there is a Fourier expansion of $F(z)$ in the variable

$$t_k = e(A_k z/\lambda_k) = e(-1/\lambda_k(z - p_k)), \tag{3.1}$$

with the A_k and λ_k of (2.1), (2.2). This is obtained as follows.

Set

$$z' = S_k z, \qquad Z = A_k z, \qquad Z' = A_k z',$$

S_k being the generator of the subgroup Γ_k of Γ which leaves p_k fixed (see (2.2)). Then (see (2.4))

$$Z' = A_k S_k z = A_k S_k A_k^{-1} Z = Z - \lambda_k.$$

Let $F(z) = g(Z)$, $F(z') = g(Z')$, and

$$\varepsilon(S_k) = e(-\alpha_k), \tag{3.2}$$

where $0 \leq \alpha_k < 1$, and note that

$$\lambda_k z + 1 - \lambda_k p_k = \lambda_k(A_k^{-1} Z) + 1 - \lambda_k p_k = \lambda_k(p_k - Z^{-1}) + 1 - \lambda_k p_k = (Z - \lambda_k)/Z.$$

Apply the transformation formula (1.1) to $F(z)$ with $z \to S_k z$, and obtain

$$\phi(Z - \lambda_k) = \phi(Z),$$

with

$$\phi(Z) = e(-\alpha_k Z/\lambda_k) Z^r g(Z).$$

Hence, $\phi(Z)$ has a Fourier series in the variable $e(Z/\lambda_k)$. Transforming back to z, we get the desired expansion of $F(z)$:

$$F(z) = (A_k z)^{-r} t_k^{\alpha_k} f_k(t_k), \qquad f_k(t) = \sum_{m=-\mu_k}^{\infty} a_m^{(k)} t^m \qquad (|t| < 1), \tag{3.3}$$

the series converging in $|t| < 1$. In accordance with the definition of Section 1, we have made the explicit assumption that the Fourier series of $F(z)$ contains only a finite number of terms with negative exponents. That is, $f_k(t)$ is a regular function of t in $|t| < 1$, except possibly (if $\mu_k > 0$) for a pole of order μ_k at $t = 0$. The finite sum

$$a_{-\mu}^{(k)} t^{-\mu} + \cdots + a_{-1}^{(k)} t^{-1} \tag{3.4}$$

is the principal part of $f_k(t)$ at $t = 0$.

We shall need the transformation formula (1.1) expressed in terms of f rather than F. Let $V \in \Gamma$ be such that $A_j V A_k^{-1}$ does not have ∞ as a fixed point ($j, k = 1, 2, \cdots$). Then, with $p_j, p_k \neq \infty$, we have, by (3.3),

$$F(Vz) = (A_j Vz)^{-r} e(\alpha_j A_j Vz/\lambda_j) f_j(e\{A_j Vz/\lambda_j\}).$$

But $F(Vz)$ is also equal, by the transformation equation (1.1), to

$$\varepsilon(V)\, m(V, z)\, F(z) = \varepsilon(V)\, m(V, z)\, (A_k z)^{-r} e(\alpha_k A_k z/\lambda_k) f_k(e\{A_k z/\lambda_k\}).$$

Writing

(3.5) $$w = A_k z, \qquad z = A_k^{-1} w, \qquad A_j V z = A_j V A_k^{-1} w = w',$$

we get

$$e(\alpha_k w/\lambda_k) f_k(e\{w/\lambda_k\}) = \varepsilon^{-1}(V)\, m^{-1}(V, A_k^{-1} w)\, w^r w'^{-r} e(\alpha_j w'/\lambda_j) f_j(e\{w'/\lambda_j\}).$$

A little algebra shows that

$$m^{-1}(V, A_k^{-1} w)\, w^r w'^{-r} = m^{-1}(A_j V A_k^{-1}, w)$$

where, however, careful note must be taken of the convention (1.2). Hence, finally,

(3.6) $$f_k(e\{w/\lambda_k\}) = \varepsilon^{-1}(V)\, m^{-1}(A_j V A_k^{-1}, w)\, e(\alpha_j w'/\lambda_j - \alpha_k w/\lambda_k) f_j(e\{w'/\lambda_j\}) \qquad (j, k = 1, 2, \cdots),$$

where w' is defined in (3.5), and V is such that $A_j V A_k^{-1}$ does not preserve ∞. This is the desired form of the transformation equation.

We shall also need the form of this equation with $A_j = I$. Introduce the expansion of $F(z)$ about ∞:

(3.7) $$F(z) = e(\alpha_0 z/\lambda_0) f_0(t), \qquad f_0(t) = \sum_{m=-\mu_0}^{\infty} a_m^{(0)} t^m, \qquad t = e(z/\lambda_0),$$

where

$$e(\alpha_0) = \varepsilon(S_0)\, e(-r) \qquad (0 \leq \alpha_0 < 1) \qquad \text{and} \qquad S_0 = \begin{pmatrix} 1 & \lambda_0 \\ 0 & 1 \end{pmatrix}.$$

Carrying through the details of the above discussion, we again get (3.6), but with $w' = V A_k^{-1} w$. By allowing a factor η of absolute value one, we can state the result for all $V \in \Gamma$:

(3.8) $$f_k(e\{w/\lambda_k\}) = \eta\, \varepsilon^{-1}(V)\, m^{-1}(V A_k^{-1}, w)\, e(\alpha_0 w'/\lambda_0 - \alpha_k w/\lambda_k) f_0(e\{w'/\lambda_0\}) \qquad (k = 1, 2, \cdots;\ |\eta| = 1).$$

4. PATH OF INTEGRATION

Let p_k be a finite parabolic point of Γ. In the subsequent discussion, we shall keep k fixed, and we shall often suppress k in the notation. We wish to find convergent series representations of the Fourier coefficients $a_m^{(k)}$ $(m \geq 0)$ of (3.3). For this purpose, we apply Cauchy's theorem to the function $f_k(t)$, which is regular in $|t| < 1$ except for a possible pole at $t = 0$:

(4.1) $$a_m^{(k)} = \frac{1}{2\pi i} \int_C f_k(t)\, t^{-m-1}\, dt,$$

where C is a circle of radius $\exp(-2\pi/N^2 \lambda_k)$ with center at the origin. $N > 0$ is arbitrary, but later we shall make $N \to \infty$. With the change of variable $t = e(w/\lambda_k)$, $w = x + iy$, we get

$$\lambda_k a_m^{(k)} = \int_L f_k(e(w/\lambda_k))\, e(-mw/\lambda_k)\, dw, \tag{4.2}$$

where L is the line segment

$$L_k(N) = L: \quad 0 \leq x < \lambda_k, \quad y = N^{-2}. \tag{4.3}$$

We must now prepare a partition of the path L which is suited to the purpose at hand. Let us remember the correspondence of the w- and $t = e(w/\lambda_k)$-planes: as $w \to i\infty$, $t \to 0$. As $N \to \infty$, L will approach the real axis or $|t| \to 1$, that is, t will tend to the boundary of the domain of existence of $f_k(t)$. In order to obtain a sharp asymptotic estimate for $f_k(t)$, we apply the transformation equation (3.6) by which the point $w \in L$ is carried into the point $w' = A_j V A_k^{-1} w$. We want w' to be "near" $i\infty$ so that $t' = e(w'/\lambda_j)$ will be near 0, and the value of $f_j(t')$ will be approximated with great accuracy by the principal part of f_j (see (3.4)). We are, therefore, led to consider the sets

$$I_{jk}(V) = I_j(V) = \{w \in L \mid T_{jk} w \in \operatorname{Int} E\} \qquad (j = 1, 2, \cdots),$$

where we have set

$$T_{jk} = A_j V A_k^{-1} = \begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_{jk} \end{pmatrix}, \qquad E = \{z \mid \Im z \geq h\}, \tag{4.4}$$

and where $h > 0$ has still to be chosen. Int E denotes the interior of the set E.

We have defined sets $I_j(V)$ corresponding to each parabolic point p_j (and certain $V \in \Gamma$). But if p_j is equivalent to p_ℓ under Γ, the expansions at p_j and p_ℓ are not essentially different. Let R_0 be a fixed fundamental region of Γ which does not contain ∞ as a vertex, and let $p_1, p_2, \cdots, p_s$ be a complete set of *inequivalent* parabolic vertices of R_0. Then we define

$$I_{jk}(V) = I_j(V) = \{w \in L \mid T_{jk} w \in \operatorname{Int} E\} \qquad (j, k = 1, 2, \cdots, s), \tag{4.5}$$

with the T_{jk} and E of (4.4).

First, restrict h by the condition $h > N^{-2}$. Then, if $I_j(V)$ is not empty, we shall have $c_{jk} \neq 0$ in T_{jk}. For $c_{jk} = 0$ implies that T_{jk} fixes ∞, that is, $V(p_k) = p_j$. But this is possible only if $j = k$, since p_k, p_j are inequivalent under Γ, and in that case $V = S_j^m$ for some integer m. Thus $T_{jk} = A_j S_j^m A_j^{-1} = U^{-m\lambda_j}$ by (2.4), or T_{jk} is a horizontal translation. Since $h > N^{-2}$, and $L = L_N$ is at height N^{-2}, T_{jk} cannot send any point of L into E and $I_j(V)$ is empty.

Consider the set $M'_{jk}(N) = M'_j$ of those V for which $I_j(V)$ is not empty. $V \in M'_{jk}$ if and only if $T_{jk}^{-1}(E)$ intersects $L = L_N$ in a nonempty interval. As we have just seen, $c_{jk} \neq 0$. Hence $T_{jk}^{-1}(E)$ is a disk of diameter $1/c_{jk}^2 h$ tangent to the real axis at $-d_{jk}/c_{jk}$. It follows that $V \in M'_j$ if and only if the equation

$$(x + d_{jk}/c_{jk})^2 + (N^{-2} - 1/2c_{jk}^2 h) = 1/4c_{jk}^4 h^2$$

possesses two solutions $x + iN^{-2}$ (x real), at least one of which lies in L. This gives the conditions

$$(4.6)\qquad 0 < c_{jk} < N/h^{1/2}, \quad -\kappa/N \le -d_{jk}/c_{jk} < \lambda_k + \kappa/N, \quad \kappa = (1/c_{jk}^2 h - N^{-2})^{1/2}$$

as a characterization of $M_j^!$. Note that

$$(4.6a)\qquad 0 \le \kappa < 1/\overline{c_{jk}}^2 h,$$

where $\overline{c_{jk}}$ is defined in (2.5).

Suppose $V \in M_j^!$. Then, for every integer m, $S_j^m V \in M_j^!$. For, by (2.4),

$$X_{jk} = A_j S_j^m V A_k^{-1} = U^{-m\lambda_j} T_{jk},$$

so that the images of a point under X_{jk} and T_{jk} are at the same height. We clearly need only one V of the class $\{S^m V\}$ in $M_j^!$. (These classes are, in fact, the right cosets of Γ_j in Γ; see (2.2).) Now

$$X_{jk} = \begin{pmatrix} a_{jk} - m\lambda_j c_{jk} & \cdot \\ c_{jk} & d_{jk} \end{pmatrix}.$$

Hence, for purposes of normalization, we can select the integer m so that $0 \le a_{jk}/c_{jk} < \lambda_j$.

As a result of these considerations, we shall define a new set $M_{jk}(N)$ as follows:

$$(4.7)\qquad \begin{aligned} M_{jk}(N) = M_j = \{V \in \Gamma \mid 0 < c_{jk} < Nh^{-1/2},\ -\kappa/N \le -d_{jk}/c_{jk} < \lambda_k + \kappa/N,\\ 0 \le a_{jk}/c_{jk} < \lambda_j\},\end{aligned}$$

where, as before,

$$A_j V A_k^{-1} = \begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_{jk} \end{pmatrix},$$

and where κ is defined by (4.6a).

Let

$$(4.8)\qquad I = \bigcup_{j=1}^{s} \bigcup_{V \in M_j} I_j(V).$$

No $I_j(V)$ is vacuous, and $I \subset L$. Moreover, it follows from (4.7) and Lemmas 1 and 2 that

(4.9) $M_j = M_{jk}(N)$ *is a finite set for each* $N > 0$.

The next step is to prove that the sets $\{I_j(V)\}$ do not overlap. Suppose $I_j(V_1) \cap I_m(V_2)$ contains an open interval ω and let

$$A_j V_1 A_k^{-1}\omega = \omega_1 \in E, \qquad A_m V_2 A_k^{-1}\omega = \omega_2 \in E.$$

Then

$$T = A_j V_1 A_k^{-1}(A_m V_2 A_k^{-1})^{-1} = A_j V_1 V_2^{-1} A_m^{-1}$$

maps ω_2 on ω_1. Let $T = \begin{pmatrix} \cdot & \cdot \\ \gamma & \cdot \end{pmatrix}$. If $\gamma \neq 0$, then T maps E on a circle C of diameter $\gamma^{-2}h^{-1}$ tangent to the real axis. Now by (2.5), $|\gamma| \geq \overline{c_{jm}}$; hence,

$$\gamma^{-2} h^{-1} \leq \overline{c_{jm}}^{-2} h^{-1}.$$

If we finally choose

$$h = \max_{1 \leq j,k \leq s} 1/\overline{c_{jk}}, \tag{4.10}$$

the diameter of C will be not larger than h, and so T maps the interior of E into its exterior. Thus T cannot map ω_2 on ω_1, since both sets lie in E. Hence, the assumption that $\gamma \neq 0$ is untenable, and T has ∞ as a fixed point. It follows that

$$V_2^{-1}(p_m) = V_1^{-1}(p_j).$$

But p_j and p_m can be equivalent under Γ only if $j = m$.

We now have $V_1 V_2^{-1}(p_j) = p_j$, so that $V_1 = S_j^m V_2$. This means that V_1 and V_2 belong to the same coset of Γ_j in Γ and therefore cannot both be in M. This completes the proof of the fact that *the sets in the partition* (4.8) *of* I *do not intersect* (*except possibly in an endpoint*).

In general, the intervals $\{I_j(V)\}$ of the above partition of I do not exhaust L. Let

$$I_0 = L - I. \tag{4.11}$$

Then, for any j, k and any $V \in \Gamma$, $A_j V A_k^{-1}(I_0)$ lies below E.

Consider the disks $D_j = A_j^{-1}(E)$ $(1 \leq j \leq s)$; D_j has diameter h^{-1} and is tangent to the real axis at p_j. Adjoin to this set the disks $D_{s+1}, D_{s+2}, \cdots, D_t$ of diameter h^{-1} tangent to the real axis at $p_{s+1}, \cdots, p_t$, the remaining parabolic cusps of the fundamental region R_0. Call D_0 the part of R_0 exterior to all D_j $(1 \leq j \leq t)$; D_0 may be empty.

Each point w of I_0 is mapped into R_0 by some VA_k^{-1} $(V \in \Gamma)$. The image of w is actually in D_0, since if it were in $D_{j'}$ for some j' $(1 \leq j' \leq t)$, then there would be an integer j in $1 \leq j \leq s$ and an element $V_1 \in \Gamma$ such that p_j is equivalent to $p_{j'}$ and $A_j V_1 A_k^{-1} w$ is in E. Then w would be in $I_j(V_1)$. Hence, define

$$I_0(V) = \{ w \in L \mid VA_k^{-1} w \in D_0 \}. \tag{4.12}$$

This time, $I_0(V)$ is either an interval or the union of a finite set of intervals; for D_0 is bounded by a finite number of arcs. From (4.8), (4.11), and (4.12), we obtain the desired partition of L into nonoverlapping sets:

$$L = \bigcup_{j=0}^{s} \bigcup_{V \in M_j} I_j(V), \tag{4.13}$$

$M_{0,k}(N) = M_0$ being the set of V on which $I_0(V)$ is not empty. Since the sets of the partition do not intersect, we have

$$(4.14) \qquad \sum_{j=0}^{s} \sum_{V \in M_j} |I_j(V)| = \lambda_k,$$

where $|I_j(V)|$ is the measure of $I_j(V)$.

Recalling that R_0 does not contain the point ∞, we note for later reference this corollary of the preceding discussion:

(4.15) *the region* D_0 *lies between two horizontal lines at heights* h_0, h_1 $(h_1 > h_0)$ *above the real axis.*

5. INTEGRATION

Going back to (4.2), we now have

$$(5.1) \quad \lambda_k a_m^{(k)} = \sum_{j=1}^{s} \sum_{V \in M_j} \int_{I_j(V)} f_k(e(w/\lambda_k))\, e(-mw/\lambda_k)\, dw + \sum_{V \in M_0} \int_{I_0(V)} = T_1 + T_2,$$

the integrands in all integrals being the same. T_2 is negligible, as we now show.

In the integral over $I_0(V)$, we apply the transformation equation (3.8) with $w \to Vw$, obtaining

$$(5.2) \quad T_2 = \sum_{V \in M_0} \varepsilon^{-1}(V)\eta \int_{I_0(V)} m^{-1}(VA_k^{-1}, w)\, e(\alpha_0 w'/\lambda_0 - m_k w)\, f_0(w'/\lambda_0)\, dw,$$

where m_k has been introduced from (1.6). When $w \in I_0(V)$, $w' = VA_k^{-1}w \in D_0$. Call D_0/λ_0 the region obtained from D_0 by applying the contraction $u \to u/\lambda_0$. Since, by (4.15), D_0/λ_0 is a compact region containing no singularities of f_0, we conclude that

$$|f_0(w'/\lambda_0)| \leq B \qquad (w \in I_0(V)).$$

Also, with $VA_k^{-1} = \begin{pmatrix} \cdot & \cdot \\ \gamma & \delta \end{pmatrix}$, we have, again from (4.15),

$$|\gamma w + \delta|^2 = \Im w / \Im w' \leq N^{-2} h_0^{-1},$$

and $\Im w' \leq h_1$, so that

$$|m^{-1}(VA_k^{-1}, w)| \leq CN^{-r}.$$

Here C denotes a general positive constant which is independent of m and N but may depend on any of the other parameters. Using these results in (5.2), we deduce that

$$|T_2| \leq \sum_{V \in M_0} |I_0(V)|\, CN^{-r} \exp(Cm/N^2).$$

The sum of the lengths of the intervals in $I_0(V)$ does not exceed λ_k, by (4.14); hence,

$$T_2 = E(m, N), \tag{5.3}$$

where we have introduced the error term

$$E(m, N) = O(N^{-r} \exp(Cm/N^2)).$$

We now treat T_1. Since in M_{jk} no $A_j V A_k^{-1}$ has ∞ as a fixed point, we can, in each integral of T_1, apply the transformation formula (3.6). In the result, replace f_j by its Fourier series (3.3), separating the principal part:

$$\begin{aligned} T_1 &= \sum_{j=1}^{s} \varepsilon^{-1}(V) \sum_{V\in M_j} \sum_{n=-\mu_j}^{-1} a_n^{(j)} \int_{I_j(V)} m^{-1}(A_j V A_k^{-1}, w)\, e\{(n+\alpha_j) w'/\lambda_j - m_k w\}\, dw \\ &+ \sum_{j=1}^{s} \varepsilon^{-1}(V) \sum_{V\in M_j} \int_{I_j(V)} m^{-1}(A_j V A_k^{-1}, w) \sum_{n=0}^{\infty} a_n^{(j)} e\{(n+\alpha_j) w'/\lambda_j - m_k w\}\, dw \\ &= T_{11} + T_{12}, \end{aligned} \tag{5.4}$$

where we have interchanged the order of summation and integration in the finite sum. (If, for a certain j, F(z) is regular at $z = p_j$, the corresponding sum in T_{11} for that j does not appear, that is, we set $a_{-1}^{(j)}, \cdots, a_{-\mu_j}^{(j)} = 0$.) We estimate T_{12}.

On $I_j(V)$, we have, by (4.4), (4.5),

$$\Im w' \geq h, \qquad |c_{jk} w + d_{jk}|^2 = \Im w / \Im w' \leq N^{-2} h^{-1},$$

and therefore

$$|m^{-1}(A_j V A_k^{-1}, w)| \leq C N^{-r}.$$

With these estimates, we can bound T_{12} by

$$|T_{12}| \leq \sum_{j=1}^{s} \sum_{V\in M_j} |I_j(V)|\, C N^{-r} \exp(Cm/N^2) \sum_{n=0} |a_n^{(j)}| \exp(-Cnh).$$

Since $h > 0$, the infinite series converges (see (3.3)). The finite sum over $|I_j(V)|$ is not more than λ_k by (4.14). Hence,

$$T_{12} = E(m, N). \tag{5.5}$$

In T_{11}, we replace the path $I_j(V)$ by the upper arc $K_j^+(V)$ of the circle $K_j(V)$ which is tangent to the real axis and passes through the endpoints of $I_j(V)$. The circle $K_j(V)$ is, in fact, the inverse image of the boundary of E under $A_j V A_k^{-1}$. Denote by $K_j^-(V)$ the lower arc of $K_j(V)$.

A single integral of T_{11} can be written

$$\int_{I_j} = -\int_{K_j} + \int_{K_j^-}, \tag{5.6}$$

where the integration is always in the positive sense. When $w \in K_j$, we have

$$\Im w' = h, \qquad |c_{jk} w + d_{jk}|^2 = yh^{-1},$$

so that $|m^{-1}(A_j V A_k^{-1}, w)| \leq C y^{r/2}$. These estimates, which hold *a fortiori* on K_j^-, yield

$$\left| \int_{K_j^-} \right| \leq |K_j^-| C N^{-r} \exp(Ch + Cm/N^2). \tag{5.7}$$

We have to sum these integrals over $V \in M_j$ and then over j.

We break the set $M_j = M_{jk}(N)$ into two parts,

$$M_{jk}(N) = M_{jk}^{(1)}(N) \cup M_{jk}^{(2)}(N),$$

where (see (4.7))

$$M_{jk}^{(1)}(N) = \{V \in M_{jk}(N) \mid 0 < c_{jk} < 2^{-1} h^{-1/2} N\}.$$

Now K_j is a circle of diameter $c_{jk}^{-2} h^{-1}$, and K_j^- is an arc which subtends the chord I_j at height N^{-2}. Since, in $M_j^{(1)}(N)$, we have $N^{-2} < 4^{-1} c_{jk}^{-2} h^{-1}$, that is, the subtended chord lies below the diameter through the center of the circle, it follows that the ratio of arc length to chord length is bounded by an absolute constant A. Hence

$$\sum_{j=1}^{s} \sum_{V \in M_{jk}^{(1)}(N)} |K_j^-| \leq A \sum_{j=1}^{s} \sum |I_j| \leq A\lambda_k = C. \tag{5.8}$$

On the other hand, in $M_{jk}^{(2)}(N)$ we have

$$2^{-1} h^{-1/2} N \leq c_j < h^{-1/2} N.$$

Let Λ be the number of circles K_j in $M_{jk}^{(2)}(N)$. Since each K_j is tangent to the real axis at a point in the interval $(0, \lambda_k)$, and the diameter of the largest circle is $4/N^2$, these circles lie in a region of area less than $4\lambda_k N^{-2} + 4\pi N^{-4} = CN^{-2}$.

A simple discussion shows that the $K_j(V)$ do not overlap. The condition for non-overlapping of two circles is that the distance between their centers should be not less than the sum of their radii; this works out in the present case to

$$|d_{jk} c'_{jk} - c_{jk} d'_{jk}| \geq h^{-1}. \tag{*}$$

Since

$$\begin{pmatrix} \cdot & \cdot \\ c_{jk} & d_{jk} \end{pmatrix} \begin{pmatrix} -d'_{jk} & \cdot \\ c'_{jk} & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ d_{jk} c'_{jk} - c_{jk} d'_{jk} & \cdot \end{pmatrix} \in A_j \Gamma A_k^{-1},$$

it follows from (2.5) that

$$|d_{jk} c'_{jk} - c_{jk} d'_{jk}| \geq \overline{c_{jk}}.$$

But $\overline{c_{jk}} \geq h^{-1}$ by (4.10), so that the condition (*) is satisfied, and the circles $K_j(V)$ $(V \in M_j)$ do not overlap.

Therefore, we have

$$4^{-1}\pi \Lambda N^{-4} < CN^{-2},$$

for the diameter of K_j must exceed N^{-2}. The total length of the circles $K_j(V)$ is, however, less than $4\pi\Lambda N^{-2}$, and this is now less than C. *A fortiori*,

$$\sum_{j=1}^{s} \sum_{V \in M_{jk}^{(2)}(N)} |K_j^-| \leq C,$$

and combining this with (5.8), we get

$$\sum_{j=1}^{s} \sum_{V \in M_{jk}(N)} |K_j^-(V)| \leq C. \tag{5.9}$$

Going back to (5.7), we now obtain from (5.9) the result that

$$\left| \sum_{j=1}^{s} \sum_{V \in M_{jk}(N)} \int_{K_j^-(V)} \right| = E(m, N). \tag{5.10}$$

6. THE MAIN THEOREMS

If we now compare (5.1), (5.3) to (5.6), and (5.10), we find that

$$\lambda_k a_m^{(k)} = - \sum_{j=1}^{s} \sum_{V \in M_{jk}(N)} \varepsilon^{-1}(V) \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \int_{K_j(V)} m^{-1}(A_j V A_k^{-1}, w) \tag{6.1}$$

$$\times\, e\{-(\nu_j w' + m_k w)\}\, dw + E(m, N),$$

where we have replaced n by $-\nu$. The quantity ν_j is defined in (1.6).

The evaluation in closed form of the integral over K_j is indeed easy. Make the substitution

$$w + d_{jk}/c_{jk} = i/p,$$

and recall that

$$w' = \frac{a_{jk}}{c_{jk}} - \frac{1}{c_{jk}^2} \frac{1}{w + d_{jk}/c_{jk}} = \frac{a_{jk}}{c_{jk}} - \frac{p}{ic_{jk}^2}.$$

Then

$$\int_{K_j} = i(ic_{jk})^r\, e\{(m_k d_{jk} - \nu_j a_{jk})/c_{jk}\} \int_{s-i\infty}^{s+i\infty} p^{-r-2} \exp\{2\pi(p\nu_j/c_{jk}^2 + m_k/p)\}\, dp,$$

where $s = c_{jk}^2 h > 0$. The integral is obviously an inverse Laplace transform, and reference to [1, p. 245] yields the value $2\pi i c_{jk}^{-r-1} L(c_{jk}, m_k, \nu_j, r)$, L being defined in (1.5). Putting this into (6.1), we get, after an interchange of order in the finite sums,

$$\text{(6.2)} \qquad \lambda_k a_m^{(k)} = 2\pi i^r \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{V \in M_{jk}(N)} \varepsilon^{-1}(V)\, e\{(m_k d_{jk} - \nu_j a_{jk})/c_{jk}\}$$

$$\times c_{jk}^{-1} L(c_{jk}, m_k, \nu_j, r) + E(m, N).$$

We now suppose $r > 0$. Keep m fixed and let $N \to \infty$. Then $E(m, N) \to 0$. The finite sum in the right member of (6.2) becomes an infinite series which, in consequence, converges, since the left member is independent of N.

In order to simplify this infinite series, we note from (4.7) that the set $M_{jk}(N)$ tends to a limit set M_{jk} as $N \to \infty$. In view of (4.6a), we see that the restriction on d_{jk} in M_{jk} is $0 \le -d_{jk}/c_{jk} \le \lambda_k$. It can actually happen that

$$V_1 = \begin{pmatrix} \cdot & \cdot \\ c_{jk} & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} \cdot & \cdot \\ c_{jk} & -\lambda_k \end{pmatrix},$$

corresponding to $-d_{jk} = 0$ and $-d_{jk} = \lambda_k$, occur in M_{jk}. In this case we make the following agreement: replace the two sets $I_{jk}(V_1)$, $I_{jk}(V_2)$ by the set

$$I_{jk}^*(V_1) = I_{ij}(V_1) \cup (U^{-\lambda_k} I_{jk}(v_2)).$$

There is no difficulty in verifying that $A_j V_1 A_k^{-1}$ maps $I_{jk}^*(V_1)$ into E. This change does not affect the value of the integral (4.2), since the integrand is periodic with period λ_k. Hence, we have, as a characterization of M_{jk},

$$M_{jk} = \{V \in \Gamma \mid 0 < c_{jk},\ 0 \le -d_{jk}/c_{jk} < \lambda_k,\ 0 \le a_{jk}/c_{jk} < \lambda_j\}.$$

This gives

$$\text{(6.3)} \qquad \sum_{V \in M_{jk}} = \sum_{\substack{c_{jk} \in C_{jk} \\ c_{jk} > 0}} \sum_{d_{jk} \in D(c_{jk})}$$

We can now utilize the definition (1.7) as a means of carrying out the summation over d_{jk}, and thus we obtain

$$\lambda_k a_m^{(k)} = 2\pi e(r/4) \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{\substack{c_{jk} \in C_{jk} \\ c_{jk} > 0}} c_{jk}^{-1} A(c_j, \nu_j, m_k) L(c_j, m_k, \nu, r).$$

This is Theorem 1.

Suppose $r = 0$. Choose $N = \beta\sqrt{m}$, where β = constant. Then $E(m, N) = O(1)$, and we get Theorem 2.

Remark. In the sum of (6.2), a_{jk} was restricted by the condition $0 \leq a_{jk}/c_{jk} < \lambda_j$ (as per (4.7)), whereas the definition (1.7) of $A(c_j, v_j, m_k)$ does not mention any condition on a_{jk}. Actually, the value of A is independent of the particular choice of a_{jk}. For a given c_{jk}, d_{jk}, all possible values of a_{jk} are contained in the set $a_{jk} - m\lambda_j c_{jk}$, where m is an integer, corresponding to the choice $S_j^m V$ instead of V. If we replace a_{jk} by $a_{jk} - m\lambda_j c_{jk}$, we see that $e\{(m_k d_{jk} - v_j a_{jk})/c_{jk}\}$ picks up the factor $e(-\alpha_j m)$. On the other hand, by the consistency condition (1.3),

$$\varepsilon^{-1}(V) \to \varepsilon^{-1}(S_j^m V) = \varepsilon^{-1}(V)\,\varepsilon^{-1}(S_j{}^m) = \varepsilon^{-1}(V)\,e(\alpha_j m).$$

Hence, the summands in A are invariant under the replacement, and therefore so is A.

7. CORRECTION

We take this opportunity to correct some errors which occur in our previous paper [3]. Reference to formulas in that paper will be indicated by an asterisk, thus: (2.4)*.

First, the definition of $L_c(m, \nu, r, \alpha)$ in (1.5)*, (1.6)* holds only for $r > 0$. For $r < -2$, define

$$L_c(m, \nu, r, \alpha) = \left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} I_{-r-1}\left(\frac{4\pi}{c\lambda}(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}\right).$$

This change affects the statement of Theorem 5, which involves the coefficients a_m of (1.7)*, and this formula, in turn, involves L_c.

Moreover, the proof of Theorem 5 suffers from a serious omission: the estimate (7.3)* of the coefficients a_m is not actually proved. (In (7.3)*, change the exponent of $(m + \alpha)$ to read $-3/4 - r/2$.) This estimate is crucially needed later to establish the existence of the functions $G_{\nu,\alpha}$ (see (7.5)*), on which the proof of Theorem 5 rests. Here we shall present a derivation of (7.3)*.

Let $u = 4\pi(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}/\lambda$, and let $C' = \{c \in C,\ c > 0\}$. Consider the series

$$P_\nu(m) = \sum_{c \in C'} c^{-1} A_{c,\nu}(m)\, L_c(m, \nu, r, \alpha) = \sum_{c<u} + \sum_{c \geq u} = S_1 + S_2. \tag{7.1}$$

Now by (1.4)*

$$|A_{c,\nu}(m)| \leq \sum_{d \in D_c} 1 = \phi(c), \text{ say.} \tag{7.2}$$

Also, $|I_{-r-1}(w)| \leq K|w|^{-r-1}$ ($|w| \leq 1$), where K denotes a general constant which does not depend on m. Hence,

$$|S_2| \leq K \sum_{c<u} c^{-1}\phi(c)\, c^{r+1}(m + \alpha)^{-r-1} < Km^{-r-1} \sum_{c \in C'} c^r \phi(c).$$

According to Poincaré (Acta Math. 1 (1882), pp. 201-206), the series

$$\sum_{c,d} |cz + d|^{\beta},$$

where (c, d) runs over the lower row of the matrices $V \in \Gamma$, converges if $\beta < -2$. Poincaré proved this result for the case where Γ has the unit circle as principal circle, but his proof can be modified to cover the present case where the principal circle is the real axis. Since $r < -2$, this gives

$$\infty > \sum_{c,d} |ci + d|^{r} > \sum_{c\in C'} \sum_{d\in D_c} c^{r} |i + d/c|^{r} > (1 + \lambda)^{r} \sum_{c\in C'} c^{r} \sum_{d\in D_c} 1 = K \sum_{c\in C'} c^{r}\phi(c).$$

Applying this result in (7.3), we get

$$|S_2| \leq Km^{-r-1}. \tag{7.4}$$

In the sum S_1, let the values of c be $1 < c_1 < c_2 < \cdots$. Then

$$S_1 = A_{1,\nu}(m)\, L_1(m, \nu, r, \alpha) + S_1',$$

where in S_1' the summation is extended over $c_1 \leq c < u$. Employing the asymptotic formula

$$I_{-r-1}(z) \sim e^{z}/\sqrt{2\pi z} \qquad (z > 1), \tag{7.5}$$

we obtain

$$|S_1'| \leq Km^{-r/2-3/4} \sum_{c_1\leq c<u} \phi(c)c^{-1/2} \exp\{4\pi(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}/c_1\lambda\}.$$

From (6.3)* we have, with $N = 2u/h$,

$$\sum_{(c,d)\in M_1} c^{-1} = \sum_{c<u} c^{-1} \sum_{d\in D_c} = \sum_{c<u} c^{-1}\phi(c) < 2u\lambda h^{-1/2} = Ku.$$

Hence,

$$\sum_{c_1\leq c<u} \phi(c)\, c^{-1/2} < \sum_{c<u} \phi(c)c^{-1/2} = \sum_{c<u} c^{1/2}\phi(c)c^{-1}$$

$$< u^{1/2} \sum_{c<u} \phi(c)c^{-1} < u^{1/2}Ku = Km^{3/4},$$

so that

$$|S_1'| \leq Km^{-r/2-3/4} \exp\{4\pi(\nu - \alpha)^{1/2} (m + \alpha)^{1/2}/c_1\lambda\}. \tag{7.6}$$

If $A_{1,\nu}(m) \neq 0$, then the first term of S_1 is of higher order in m than S_1' or S_2. Hence, using (7.5) again, we have

(7.7) $P_\nu(m) \sim (\lambda/2\pi)^{1/2}(\nu - \alpha)^{r/2+1/4} m^{-r/2-3/4} \exp\{4\pi(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}/\lambda\}$,

and in any event

(7.8) $$P_\nu(m) < Km^{-r/2-3/4} \exp(8\pi\mu\sqrt{m}/\lambda).$$

Since, by (1.7)*, $a_m = (2\pi/\lambda)\Sigma_{\nu=1}^{\mu} a_{-\nu} P_\nu(m)$, this gives (7.3)*.

We note a number of minor errors and misprints in [3]. The notation "ℓ.3*" means line 3 from the bottom of the page.

p. 266, ℓ.2: For $0 \leq \alpha \leq 1$, read $0 \leq \alpha < 1$.

p. 267, ℓ.5: Insert the factor $(m + \alpha)$ in the sum of the right member.

p. 268, ℓ.13: For V, read U_n.

p. 273, ℓ.6*: This formula requires further explanation. What is involved here is a characterization of the set M_1. Now, by definition (see (4.5)*) and by the remark following (5.1)*, we know that $(c, d) \in M_1$ if $0 < c < Nh/2$, $0 \leq -d/c < \lambda$. (The latter condition is the same as $d \in D_c$.) What we have to show is that these are *necessary* conditions. However, the following situation can arise: there may be two pairs (c, d), $(c, d + c\lambda)$ belonging to M_1. If we define

$$I'_{c,d} = \{z \mid \Im z = y_0,\ Vz \in R\},$$

then $I'_{c,d} = S^{-1}I'_{c,d+c\lambda}$ and both I's lie partly in L_N, partly not in L_N. In such cases we make the following agreement: replace the two sets $I_{c,d}$, $I_{c,d+c\lambda}$ in the partition (3.5)* by $I^*_{c,d}$, where

$$I^*_{c,d} = I_{c,d} \cup S^{-1}I_{c,d+c\lambda}.$$

Because of the periodicity of the integrand, we have

$$\int_{I^*_{c,d}} e(-(m + \alpha)z/\lambda)\, F(z)\, dz = \int_{I_{c,d}} \cdots + \int_{I_{c,d+c\lambda}} \cdots,$$

so that the value of the integral (4.1)* is not affected. The similar situation involving (c, d) and $(c, d - c\lambda)$ is handled in an analogous fashion. With this convention, M_1 can be characterized as the set of (c, d) with $0 < c < Nh/2$, $d \in D_c$, as stated in the formula in question.

p. 274, ℓ.6*: For $(m + \alpha)^{3/4-r/2}$, read $(m + \alpha)^{-3/4-r/2}$.

ℓ.1*,2*: Delete these lines.

p. 275, ℓ.2: After $r = 0$, insert $\alpha = 0$.

ℓ.3: Note that $A_{1,1}(m)$ consists of just one term, since the only element of $\Gamma(\lambda)$ with $c = 1$, $0 \leq -d < \lambda$ is $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence, $A_{1,1}(m) = 1$.

p. 276, ℓ.7: For $c > 0$, $d \leq 0$, read $c > 0$.

ℓ.10: For $c \geq 0$, $d \leq 0$, read $c > 0$.

ℓ.11: Delete this line.

ℓ.13: For $V \in S'$, $V \neq I$, read $V \in S'$.

p. 276, ℓ.15: For $e(-(\nu - \alpha)z\lambda)$, read $e(-(\nu - \alpha)z/\lambda)$,

For $V \in S'$, $V \neq I$, read $V \in S'$.

ℓ.17: Delete this line.

ℓ.3*: For (7.11), read (7.10).

p. 277, ℓ.1: For $c \geq 0$, read $c > 0$.

ℓ.12*: For (7.14), read (7.13).

p. 278, ℓ.8: For $-2\pi t$, read $-2\pi mt$.

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