ON SUBFACTORS OF FACTORS OF TYPE II_1

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1. INTRODUCTION

In the series of papers entitled *On Rings of Operators*, Murray and von Neumann study certain classes of operator algebras on a Hilbert space $\mathcal{H}$. Among the more remarkable types of algebras are the factors of type II_1 [3, p. 172] which, although they have infinitely many orthogonal nonzero projections (self-adjoint idempotents), have a unique linear functional tr such that

(1) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$,

(2) $\mathrm{tr}(A^*A) \geq 0$, and $\mathrm{tr}(A^*A) = 0$ only if $A = 0$,

(3) $\mathrm{tr}(I) = 1$, where I is the identity operator.

An $f \in \mathcal{H}$ such that $\mathrm{tr}(A) = \alpha(Af, f)$ for $\alpha > 0$ will be called a trace vector. Although Murray and von Neumann assume $\mathcal{H}$ to be separable, subsequent work has shown this assumption to be unnecessary, and most proofs in [3], [4], [5] do not assume separability of $\mathcal{H}$. Therefore, the definitions and notation of [3] will be used, except that factors will be designated by script letters. All isomorphisms mentioned will preserve the adjoint operation.

In [5, Section 5.3] it is shown that any (countable) group G whose non-identity conjugate classes contain infinitely many elements will lead to a factor of type II_1 on a (separable) Hilbert space. In this paper, we study relationships between a II_1-factor and a II_1-subfactor which are reminiscent of group and subgroup relationships. The work was motivated by the factors generated in the manner of [5] by the group of all finite permutations of the integers and the subgroup of all even permutations.

First we select the factors to be studied. In [4, Theorem II], it is shown that a II_1-factor $\mathcal{M}$ with a vector cyclic under $\mathcal{M}$ and $\mathcal{M}'$ (we use the superscript ' to denote the commutant [3, p. 117]) possesses a trace vector $f \in \mathcal{H}$ with $\|f\| = 1$. Associated with f, which will now be fixed, is an anti-isomorphism $A \to A'$ for $A \in \mathcal{M}$, $A' \in \mathcal{M}'$ defined by $Af = A'f$. Hence, $\mathrm{tr}_{\mathcal{M}'}(A') = (A'f, f)$. The details of the anti-isomorphism are in [4, Chapter IV]. Let $\mathcal{A} \subset \mathcal{M}'$ be a II_1-subfactor such that $\mathcal{A}'$ is finite. Let

$$c_1 = \dim_{\mathcal{A}'}([\mathcal{A}f]),$$

where $[\mathcal{P}f]$ = closure of $\{Sf: S \in \mathcal{P}\}$. Let $\mathcal{N}' = \{A' \in \mathcal{M}': A'f = Af \text{ for } A \in \mathcal{A} \subset \mathcal{M}\}$. Then $\mathcal{N}'$ is anti-isomorphic to $\mathcal{A}$, and is weakly closed by [5, p. 728]. Let $c_2 = \dim_{\mathcal{N}}([\mathcal{N}'f])$. We shall show that $c_1 = c$. Since $\mathcal{A} \subset \mathcal{M}$, any trace vector for $\mathcal{M}$ will be a trace vector for $\mathcal{A}$, but $\mathcal{A}$ will have trace vectors which are not trace vectors for $\mathcal{M}$. We shall study trace vectors g for $\mathcal{A}$ which lie in the "smallest" subspaces $\eta\mathcal{M}'$ in which such trace vectors can lie, that is, in subspaces of dimension c_1 by [3, Lemma 9.3.3]. Theorem 1 shows that $g = \alpha Vf$, where $V \in \mathcal{M}$ is a partial isometry with $\dim(V^*V) = c_1$. If $c_1 = 1/n$ for integral n and there are "enough" different V's giving trace vectors for $\mathcal{A}$, then there is a coset-like decomposition of $\mathcal{H} = [\mathcal{A}f_1] \oplus \cdots \oplus [\mathcal{A}f_n]$, where f_k is a trace vector for $\mathcal{M}$. If

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$c_1 = 1/2$, there always are enough V's; and if $\mathscr{A}$, in a suitable representation, has a complete, orthonormal set of trace vectors, then the same is true of $\mathscr{M}$. This is a very special case of a conjecture of Singer.

In Section 3, we obtain a representation theorem for $\mathscr{M}$ in terms of $\mathscr{A}$. Here, too, no special hypotheses are needed if $c = 1/2$, but conditions akin to normality of subgroups are needed if $c = 1/n$ $(n > 2)$.

2. TRACE VECTORS FOR SUBFACTORS

We shall deal with the factors $\mathscr{N} \supset \mathscr{M} \supset \mathscr{A}$ and their commutants $\mathscr{N}' \subset \mathscr{M}' \subset \mathscr{A}'$. Since f is a trace vector for $\mathscr{M}$ and $\mathscr{M}'$, by [3, pp. 142-143] and [4, Lemma 4.1.2], every vector $g' \in \mathscr{H}$ can be written as

$$g' = VBf = CWf = B'V'f = W'C'f,$$

where $V, W \in \mathscr{M}$, $V', W' \in \mathscr{M}'$ are partial isometries and $B, C \,\eta\, \mathscr{M}$, $B', C' \,\eta\, \mathscr{M}'$ are possibly unbounded, densely defined and defined on f, closed by [3, Theorem XV] and self-adjoint. Therefore they have a unique resolution of unity.

THEOREM 1. *Let* g *be a trace vector for* $\mathscr{A}$ *with the property that for some projection* $E' \in \mathscr{M}'$, $\dim(E') = c_1$ *and* $E'g = g$. *Then there is a partial isometry* $V \in \mathscr{M}$ *and a scalar* $\alpha > 0$ *such that* $g = \alpha Vf$.

The proof will need a sequence of lemmas. It should be noted that every projection $E' \in \mathscr{M}'$ with $\dim(E') = c_1$ has a trace vector for $\mathscr{A}$ in the subspace $E'\mathscr{H}$. Indeed, since $\dim_{\mathscr{A}}([\mathscr{A}f]) = c_1$ and $\mathscr{M}'$ is contained in the factor $\mathscr{A}'$, there is a partial isometry $W' \in \mathscr{A}'$ [3, Theorem VII] such that $W'[\mathscr{A}f] = E'\mathscr{H}$. Clearly, $W'f$ is a trace vector for $\mathscr{A}$.

LEMMA 2.1. *Let* $A \in \mathscr{A}$; *let* V_1 *and* V_2 *be partial isometries in* $\mathscr{M}$; *and let* $B_1, B_2 \,\eta\, \mathscr{M}$ *be positive and defined on* f. *Then*

$$(AV_1B_1f, V_2B_2f) = (A'V_2'^*B_2'f, V_1'^*B_1f),$$

where $A'f = Af$ *and* $A' \in \mathscr{N}$.

Proof. By a well-known spectral theorem, $B_jf = \lim_{n\to\infty} C_{nj}f$, where $j = 1, 2$ and $C_{nj} = \int_0^n \lambda \, dE_{\lambda j}$. Since $A \to A'$ gives an anti-isomorphism of $\mathscr{M}$ and $\mathscr{M}'$, the lemma is immediate for bounded B_j. Hence

$$(AV_1B_1f, V_2B_2f) = \lim_{n\to\infty} (AV_1C_{n1}f, V_2B_2C_{n2}f)$$

$$= \lim_{n\to\infty} (A'V_2^{*\prime}C_{n2}f, V_1^{*\prime}C_{n2}f) = (A'V_2^{*\prime}B_2f, V_1^{*\prime}B_1f).$$

COROLLARY. *If* V_1B_1f *is a trace vector for* $\mathscr{A}$, *then* $V_1^{*\prime}B_1f$ *is a trace vector for* $\mathscr{N}'$.

Proof. Lemma 2.1, together with the anti-isomorphism between $\mathscr{A}$ and $\mathscr{N}'$ and the uniqueness of the trace subject to conditions (1) to (3) of Section 1, gives the corollary.

LEMMA 2.2 *If* $\mathscr{A}'$ *is finite, then* $\mathscr{N}$ *is finite and* $c_1 = c_2$.

Proof. By Lemma 2.1, the subspaces $[\mathscr{A} V_j B_j f]$ are pairwise othogonal if and only if the subspaces $[\mathscr{N}' V_j'^{*} B_j f]$ are orthogonal. Hence if there are at most q orthogonal subspaces of the form $[\mathscr{A} V_j B_j f]$, where $V_j B_j f$ is a trace vector for $\mathscr{A}$, then there are at most q orthogonal subspaces of the form $[\mathscr{N}' V_j'^{*} B_j f]$, where $V_j'^{*} B_j f$ is a trace vector for $\mathscr{N}'$ by the corollary. By [3, Chapter VII], a II_1-factor with at most finitely many orthogonal, equivalent nonzero projections is finite, and by [3, Lemma 9.3.3] all subspaces of the form $[\mathscr{N}' f']$, where f' is faithful under $\mathscr{N}'$, are equivalent under $\mathscr{N}$.

Now let $E' \in \mathscr{M}'$ with $\dim E' = c_1$. By a previous remark, there is a trace vector $g = E'g$ for $\mathscr{A}$. For suitable $V' \in \mathscr{M}'$ and $B' \,\eta\, \mathscr{M}'$, $g = V'B'f$. Since $E'g = g$, $E'V' = V'$ by [4, Chapter IV]. But $\dim_{\mathscr{A}'}([\mathscr{A} g]) = c_1 = \dim E'$, and therefore the projection on the range of V' cannot have dimension less than c_1. By the finiteness of $\mathscr{A}'$, E' is the projection on the range of V'. Dually, by the corollary, V^*Bf is a trace vector for $\mathscr{N}'$, and since $E'V' = V'$, it follows that $VE = V$ and $EV^* = V^*$. Moreover, E is the smallest projection such that $EV^* = V^*$. By [3, Lemma 6.2.1] and the fact that dimension is invariant under anti-isomorphism, E, the projection on the range of V^*, has dimension c_1. But $E\mathscr{H} \supset [\mathscr{N}' V^* Bf] \,\eta\, \mathscr{N}$, so that $c_1 \geq c_2$. By duality, $c_2 \geq c_1$.

LEMMA 2.3. *If* $g = V'B'f$ *is a trace vector for* $\mathscr{A}$, *where* V', B' *are as above,* $B'f$ *is a trace vector for* $\mathscr{A}$.

Proof. $(AV'B'f, V'B'f) = (AB'f, V'^{*}V'B'f) = (AB'f, B'f)$ by the structure of the canonical (polar) decomposition of an operator $\eta\, \mathscr{M}'$.

Proof of the theorem. Let $E_1', E_2', \cdots, E_q' \in \mathscr{M}'$ be orthogonal projections of dimension $\leq c_1$ such that $E_1' + \cdots + E_q' = I$. (We can choose $q = [c_1^{-1}] + 1$ for definiteness). By [4, pp. 234-5] and previous remarks, there exist vectors $f_1, \cdots, f_q$ with $E_j' f_j = f_j$ and $\mathrm{tr}_{\mathscr{A}'}(A') = \Sigma_{j=1}^{q} (A'f_j, f_j)$. But $f_j = V_j B_j f$ as above, and since $E_j' f_j = f_j$, it follows that $E_j' V_j B_j f = V_j B_j f$. By the cyclicity of f under $\mathscr{M}$ and $\mathscr{M}'$, and since $E_j' \in \mathscr{M}'$, $B_j(I - E_j) = 0$. By Lemma 2.3 and [4, Theorem III], we can choose $f_1 = B_1 f$ with $\|f_1\|^2 = c_1$ to be a trace vector for $\mathscr{A}$, since $E_1' \mathscr{A}' E_1'$ is anti-isomorphic to $\mathscr{A}$ when $\dim(E_1') = c_1$. Now by the orthogonality of the E_j' and the consequent orthogonality of their anti-isomorphic images E_j, the closure of $B_j B_k$ is 0 whenever $j \neq k$. Therefore, for any $M' \in \mathscr{M}'$ and $g = \Sigma_{j=1}^{q} B_j f$, a computation shows that

$$\mathrm{tr}_{\mathscr{M}'}(M') = \mathrm{tr}_{\mathscr{A}'}(M') = \sum_{j=1}^{q} (M'V_j B_j f, V_j B_j f) = \sum_{j=1}^{q} (M' B_j f, B_j f) = (M'g, g).$$

Similarly, using $g_j = V_j^{*\prime} B_j f$ instead of f_j and $M \in \mathscr{M}$ instead of M', we see that $\mathrm{tr}_{\mathscr{M}}(M) = (Mg, g)$, since $B_j' f = B_j f$. Hence g is a joint trace vector for $\mathscr{M}$ and $\mathscr{M}'$, and by [4, Lemma 4.2.3.], it must be of the form Uf, for some unitary $U \in \mathscr{M}$. Hence each B_j has a bounded extension which is positive, and $(\Sigma B_j)^2 = I$. This implies that $B_j = E_j$, and in particular that $B_1 = E_1$. Returning to any $g = E'g$ which is a trace vector for $\mathscr{A}$ with $\dim E' = c_1$, we have $g = V_1' E_1 f = E_1 V_1 f$ up to a normalization. But E_1 is the projection on the range of V_1, so that $g = V_1 f$ except for normalization.

The proof of Theorem 1 shows that there is always a projection $E \in \mathscr{M}$ with $\dim(E) = c_1$ such that Ef is a trace vector for $\mathscr{A}$. If $c_1 = 1/n$ for some positive integer n, we can ask whether there exist n orthogonal projections $E_1, \cdots, E_n$ such that $E_j f$ is a trace vector for $\mathscr{A}$ $(j = 1, 2, \cdots, n)$. In the case where $c_1 = 1/2$, it is clear that if Ef is a trace vector for $\mathscr{A}$, then $(I - E)f$ is also such a trace vector. A fallacious induction led the author to assert in [2] the existence of such projections for $n > 2$.) In the case of general integral n, it can be shown that there is a unitary $W \in \mathscr{A}$ such that $F = WEW^* \neq E$ but Ef and Ff are both trace vectors for $\mathscr{A}$. However, it is not clear whether $EF = 0$.

With $E_1, \cdots, E_n$ as above, we set $U = \Sigma_{j=1}^{n} \omega^j E_j$, where ω is a principal n^{th} root of unity. A computation reveals that the subspaces $[\mathscr{A}U^j f]$ and $[\mathscr{A}U^k f]$ are orthogonal if $j \not\equiv k \pmod{n}$.

Let $\mathscr{H}_1 = [\mathscr{A}f]$, and let $\mathscr{A}_1$ be the restriction of $\mathscr{A}$ to $\mathscr{A}_1$. Then f is cyclic (in $\mathscr{H}_1$) under $\mathscr{A}$, and therefore, by [4, Lemma 4.2.3], there exists, corresponding to each trace vector h for $\mathscr{A}$ which is cyclic in $\mathscr{H}_1$, a unitary $V \in A$ such that $h = Vf$, up to a scalar factor. We say that a factor of type II_1 has the C.O.N. property on a Hilbert space if there exists a C.O.N. set of cyclic trace vectors for it.

THEOREM 2. *If* $c_1 = 1/n$, *if* $\mathscr{A}$ *has the* C.O.N. *property on* $\mathscr{H}_1$, *and if there are projections* $E_1, \cdots, E_n \in \mathscr{M}$ *such that the* $E_k f$ *are trace vectors for* $\mathscr{A}$, *then* $\mathscr{M}$ *has the* C.O.N. *property on* $\mathscr{H}$.

Proof. Let $\{V_\alpha f\}$, where the $V_\alpha \in \mathscr{A}$ are unitary, be a C.O.N. set of trace vectors for $\mathscr{A}$ in $\mathscr{H}_1$. Then the sets $\{V_\alpha U^k f\}$ $(k = 1, 2, \cdots, n)$ are orthonormal and span $[\mathscr{A}Uf] \oplus \cdots \oplus [\mathscr{A}U^{n-1}f] \oplus [\mathscr{A}f] = \mathscr{H}$. Since V_α and U^k are unitary, $V_\alpha U^k \in \mathscr{M}$ is unitary and $(V_\alpha U^k f, V_\beta U^j f) = \delta_{\alpha\beta}\delta_{jk}$ for $k, j = 1, 2, \cdots, n$.

3. STRUCTURE OF FACTORS

In this section we derive, under suitable hypotheses, a structure theorem for the factor $\mathscr{M}$ in terms of the subfactor $\mathscr{A}$.

LEMMA 3.1. *Let* $U \in \mathscr{M}$ *be a unitary operator such that* $UAU^*f \in [\mathscr{A}f]$ *for all* $A \in \mathscr{A}$. *Then* $UAU^* \in \mathscr{A}$.

Proof. $UAU^*f = \lim_{n\to\infty} A_n f$ for suitable $A_n \in \mathscr{A}$. Hence, by [5, p. 728], $UAU^* \in \mathscr{A}$.

It is clear that if $U^n = I$ and $UAU^*f \in [\mathscr{A}f]$ for each $A \in \mathscr{A}$, then $A \to UAU^*$ is an automorphism of $\mathscr{A}$. In the case where $c = 1/2$, the $U = E_1 - E_2$ of Section 2 satisfies $U^2 = I$ and $(Af, BUf) = 0$ for all $A, B \in \mathscr{A}$. Therefore,

$$(UAUf, BUf) = (UAf, Bf) = (UAB^*f, f) = (B^*f, A^*Uf) = 0$$

and $UAUf = UAU^*f \in [\mathscr{A}f]$.

THEOREM 3. *Let* $\mathscr{A}, \mathscr{M}, \mathscr{A}', \mathscr{M}'$, f *and* $U \in \mathscr{M}$ *be as above, with* $c_1 = 1/2$. *Then each* $M \in \mathscr{M}$ *can be written uniquely as* $M = A_1 + A_2 U$ *with* $A_1, A_2 \in \mathscr{A}$. *The multiplication is characterized by the automorphism of* $\mathscr{A}$ *given by* $A \to UAU$.

Proof. Since $c = 1/2$, $(Af, BUf) = 0$ for $A, B \in \mathscr{A}$ and $[\mathscr{A}f] \oplus [\mathscr{A}Uf] = \mathscr{H}$. Thus the vectors $A_1 f + A_2 Uf$ are dense in $\mathscr{H}$, and by [5, p. 728] the operators $A_1 + A_2 U$ are weakly dense in $\mathscr{M}$.

Since $c = 1/2$, any $B' \in \mathscr{A}'$ can be written as $B' = \Sigma_{i,j=1}^{2} E_i' A_{ij}' E_j'^*$, where $E_1' \in \mathscr{A}'$ is the projection on $[\mathscr{A}f]$, and where $E_2' \in \mathscr{A}'$ is defined by $E_2' Af = AUf$, $E_2' AUf = 0$, so that it is a partial isometry carrying $[\mathscr{A}f]$ onto $[\mathscr{A}Uf]$. The $A_{ij}' \in E_1' \mathscr{A}' E_1'$, and since $[\mathscr{A}f] = E_1' \mathscr{H}$, $A_{ij}' f = A_{ij} f$ for a unique $A_{ij} \in \mathscr{A}$. We now ascertain which $B' = \Sigma E_i' A_{ij}' E_j'^*$ are elements of $\mathscr{M}'$. For this purpose, we note that

$$B'f = E_1' A_{11}' f + E_2' A_{21}' f = A_{11} f + A_{21} Uf$$

and

$$B'Uf = E_1'A_{12}'f + E_2'A_{22}'f = A_{12}f + A_{22}Uf.$$

$B' \in \mathcal{M}'$ if and only if $B'(A + BU) = (A + BU)B'$ for each $A, B \in \mathcal{A}$, since the $A + BU$ are weakly dense in $\mathcal{M}$ and since $B' \in \mathcal{A}'$, $B'A = AB'$; therefore we need only require that $B'U = UB'$. In fact, we need only require that $B'Uf = UB'f$, since upon setting $A_1 = UAU$ and $B_1 = UBU$ we have

$$B'U(A + BU)f = B'A_1Uf + B'B_1f = A_1UB'f + B_1B'f = UAB'f + UBUB'f$$

$$= UB'Af + UB'BUf = UB'(A + BU)f.$$

By the boundedness of U and B' and the density of the $(A + BU)f$, $B'U = UB'$.

In order that $B'Uf = UB'f$, we need $A_{12}f + A_{22}Uf = UA_{11}f + UA_{21}Uf$. By the orthogonality of $[\mathcal{A}f]$ and $[\mathcal{A}Uf]$, we must therefore have $A_{12}f = UA_{21}Uf$ and $A_{22}Uf = UA_{11}f$. Hence

$$\text{(i)} \qquad A_{12} = UA_{21}U \text{ and } UA_{22}U = A_{11},$$

since f is faithful for $\mathcal{M}$.

Moreover, it is clear that if equations (i) hold, the induced B' is a member of $\mathcal{M}'$.

Now $Bf = B'f = A_{11}f + A_{21}Uf$ for $A_{11}, A_{21} \in \mathcal{A}$, and the theorem is proved.

When $c = 1/n$, we could hope for a representation of this type. Certainly the existence of a unitary $U \in \mathcal{M}$ with $U^n = I$, $U\mathcal{A}U^* \subset \mathcal{A}$, and $[\mathcal{A}U^jf]$ orthogonal to $[\mathcal{A}U^kf]$ if $j \not\equiv k \pmod n$ would allow us to carry through the above proof. However, a counterexample due to J. E. McLaughlin shows that there are factors $\mathcal{M}, \mathcal{A}$ such that for any $U \in \mathcal{M}$, $U\mathcal{A}U^* \subset \mathcal{A}$ implies $U \in \mathcal{A}$. These factors are generated as in [5] by discrete groups whose nontrivial conjugate classes are infinite. Let G be the group of 2-by-2 unimodular matrices with integral coefficients. Let G_1 be the subgroup of all matrices whose lower left-hand entry is even. Let Z be the center of G, that is, $\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right]$. Let $\overline{G} = G/Z$ and $\overline{G}_1 = G_1/Z$. Let $\mathcal{M}$ be the factor associated with $\overline{G}$, and $\mathcal{A}$ the factor associated with $\overline{G}_1$. It is known that $\overline{G}_1$ has index 3 in $\overline{G}$, so that $c = 1/3$. The proof of Lemma 3 in [1] shows that if $\overline{G}_1$ satisfies conditions 1 and 2 below, then each $A \in \mathcal{M}$ for which $A\mathcal{A}A^* \subset \mathcal{A}$ must be in $\mathcal{A}$.

Our conditions are essentially (ii) of [1], namely: if for each finite set $B \subset \overline{G}$ and for every $x \in \overline{G} - \overline{G}_1$ there is a $y \in \overline{G}_1$ such that

1. $x^{-1}yx \notin \overline{G}_1$ for all $x \in \overline{G} - \overline{G}_1$,
2. $z \in B$, $w \in B$ and $z^{-1}yw = y$ implies $z = y$.

If y is the coset containing $\begin{pmatrix}r & 1\\ r^2 - 1 & r\end{pmatrix}$ where r is an odd number much larger than the entries of the (finitely many) matrices in the cosets of B, we see that $\overline{G}$ and $\overline{G}_1$ satisfy the conditions.

4. CONCLUSION

There appear to be many open questions in this area. A characterization of the projections $E \in \mathscr{M}$ for which $\dim E = c_1$ and Ef is a trace vector for $\mathscr{A}$ would be desirable. Do there always exist n such orthogonal projections, if $c_1 = 1/n$? If not, what is the supremum of those projections?

In connection with Section 3, one can ask whether $\mathscr{M}$ is approximately finite ([5]) if $\mathscr{A}$ is so. This would be the case if one could show that there are finite-dimensional rings $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}$ such that $\mathscr{A}$ is the smallest ring of operators containing all of them, and $U\mathscr{A}_k U^* \subset \mathscr{A}_{k+p}$.

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