GROUPS ON R^n OR S^n

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1. INTRODUCTION

Throughout this paper G will be a compact connected Lie group acting on a manifold M which is either R^n or S^n, that is, euclidean n-space or the n-sphere. Furthermore the group is assumed to act differentiably, by which is meant that each homeomorphism of M is of class C^1 in the ordinary differentiable structure of M. The space M is divided into certain disjoint subsets as follows. If r is the highest dimension of any orbit, let B be the set of points on orbits of dimension less than r. The set B is closed, and it is known [1] that $\dim B \leq n - 2$. Let D be the set of points x satisfying

a) $\dim G(x) = r$,

b) in every neighborhood of x there is a point y such that G_y, the isotropy group at y, has fewer components than G_x.

Any orbit in D is called an exceptional orbit of highest dimension. Near such an orbit G(x), there is another highest-dimensional orbit G(y) which "wraps around" G(x) more than once.

Let U be the set of all points on orbits of highest dimension which are not in D. Then $x \in U$ if and only if

a) $\dim G(x) = r$,

b) for all y in some neighborhood of x, G_x and G_y have the same number of components.

The sets B, D, U are invariant and disjoint, and

$$M = B \cup D \cup U;$$

B is closed, $B \cup D$ is closed, and U is open. For the case at hand [2],

$$\dim D \leq n - 2.$$

The orbits of M can be made into a space M*, called the orbit space; and M* contains the disjoint sets B*, D*, U* which are the images of B, D, U under the map from M to M*. The map from M to M* is denoted by T.

This paper studies some of the properties of these sets, and it proves the following theorems.

THEOREM 1. *Let a compact connected Lie group* G *act differentiably on* $M = R^n$ *or* S^n. *Then* $U^* \cup D^*$ *is simply connected.*

COROLLARY. *Under the hypothesis of Theorem* 1, *let* $\dim D \leq n - 3$. *Then* U* *is simply connected.*

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THEOREM 2. *Let a compact connected Lie group* G *act differentiably on* $M = R^n$ *or* S^n. *Then* U* *is orientable.*

This implies (see Corollary 1, stated later) that the r-dimensional homology of the orbits in U forms a constant sheaf over U*.

2. DEFINITION AND STRUCTURE OF C*

Let m(x) be the number of components of G_x, and let

$$M_i = \{x, x \in M, \dim G(x) = i\},$$

$$M_{ij} = \{x; x \in M_i, m(x) = j\}.$$

Then

$$B = \bigcup_{i=0}^{r-1} M_i,$$

and it is known [2] that if k is the smallest number of components in any r-dimensional orbit, then

$$U = M_{rk}.$$

Any component of M_{ij} is a manifold. Let

$$C = \{x; x \in B, m(x|B) \text{ continuous at } x, \dim B = n - 2 \text{ at } x\},$$

and

$$E = \{x; x \in D, m(x|D) \text{ continuous at } x, \dim D = n - 2 \text{ at } x\}.$$

Note that each point of C must be in M_{r-1} [1]. Note also that it is shown later, in Lemma 4, that $U^* \cup E^*$ is simply connected, which sharpens Theorem 1.

LEMMA 1. *Under the hypothesis of Theorem 1, let* $b^* \in C^*$. *Then* M* *contains a closed* (n - r)*-cell which is a neighborhood of* b*; *the boundary of the cell contains* b* *and a neighborhood of* b* *relative to* B*. *Furthermore the cell may be so chosen as to contain no point of* D*.

For the proof, let b be a point of C such that

$$T(b) = b^*,$$

and let K be an (n - r + 1)-cell which is a slice at b [3, 5], or a pseudo-section in the terminology of Mostow. It may be assumed that G_b acts orthogonally on a neighborhood of b in M and on K. The slice K is chosen by definition to satisfy the following [3, 5]:

a) $G_b(K) = K$,

b) for any $x \in K$, $G_x \subset G_b$,

c) there is a closed (r - 1)-cell Q in G (a section of the cosets of G_b at e) such that $(g, x) \to g(x)$ gives a homeomorphism of $Q \times K$ onto a neighborhood of b in M,

d) if $g \in G - G_b$, then $K \cap g(K) = \emptyset$.

Let the fixed points of G_b be denoted by $F(G_b)$, and let

$$L_1 = F(G_b) \cap K;$$

L_1 is an $(n - r - 1)$-cell, and if L_2 is an orthogonal two-cell in K, we may assume that

$$K = L_1 \times L_2.$$

Then, for $k \in K$,

$$k = (l_1, l_2) \qquad (l_1 \in L_1, l_2 \in L_2),$$

and for any $g \in G_b$,

$$gk = (gl_1, gl_2) = (l_1, gl_2).$$

Thus the action of G_b on K is determined by its action on L_2. If N is the subgroup of G_b leaving all of K fixed, then $\dim G_b/N = 1$. There are two possibilities:

a) G_b/N is a circle which acts on L_2 as the rotation group;

b) G_b/N is a circle, extended by an element of order 2, which reverses the orientation of L_2.

However, case b) is impossible because an element which reverses orientation in L_2 would reverse orientation in M. Therefore G_b/N is a circle, and the orbits of G_b/N in L_2 are concentric circles.

In L_2 choose a segment A from the origin to the edge of L_2. This is a cross-section of the orbits in L_2. Hence $L_1 \times A$ is a cross-section of the orbits of G_b in K; it is also a cross-section of the orbits of G in G(K). Hence $L_1 \times A$ is mapped topologically by T, and the image $T(L_1 \times A)$ is the cell (note that it contains no point of D^*) whose existence is asserted in the Lemma. This completes the proof.

LEMMA 2. *Let* $M = R^n$ *or* S^n, *and let* K *be a finite* C^1*-complex in* M, $\dim K = i$. *Then* M *may be triangulated so that no* $(n - i - 1)$*-cell of* M *touches* K.

We may assume $i \leq n - 1$, since otherwise there is nothing to prove; therefore K can not be all of S^n. If we omit one point, S^n becomes R^n, and it will be assumed that $M = R^n$, which involves no loss of generality.

Assume that a triangulation of $M = R^n$ is given. It will be shown that this can be modified slightly to yield a triangulation which satisfies the conclusion. The given triangulation may be assumed to have the property that it is a triangulation after a slight alteration of the vertices [8, p. 370].

The vertices of the given triangulation are countable and may be denoted by $p_1, p_2, \cdots$. If p_1 is not on K, let $p_1' = p_1$; if p_1 is on K, let p_1' be a point not on K and sufficiently near p_1 to satisfy the remark above [8, p. 370]. Assume now that the new vertices $p_1', \cdots, p_w'$ have been selected. Choose p_{w+1}' sufficiently near p_{w+1} (in the sense above) and so that any $(n - i - 1)$-linear space spanned by any set of $(n - i)$ of the points p_j' $(j \leq w + 1)$ does not touch K. This is possible since the cells of K are of class C^1. Continuing by this inductive procedure, we obtain points p_1', p_2', $\cdots$, and the triangulation they determine has the desired properties. This completes the proof.

3. PROOF OF THEOREM 1

Let α^* be a path in $U^* \cup D^*$ with end points at p^*. It is known that α^* is covered by a path α in $U \cup D$ with end points at p, $T(p) = p^*$ [3]. By hypothesis α can be shrunk in M. This means there exists a map f into M of the unit square

$$\sigma = \{ s, t: 0 \leq s \leq 1,\ 0 \leq t \leq 1\}$$

with

1) $f(0, t)$ defining α,

2) $f(s, 0) = f(s, 1) = f(1, t) = p$.

In order to prove the theorem it will be sufficient to prove that f can be deformed so that $f(\sigma)$ does not touch B.

We shall proceed by a finite induction to show that $f(\sigma)$ may be successively freed from $M_0, M_1, \cdots, M_{r-1}$. The procedures for $M_0, \cdots, M_{r-2}$ are similar, but the procedure for M_{r-1} is different. We begin by considering M_0 and making the successive steps required to reach the case of M_{r-1}. In doing so, we assume these cases to be present; for if they were not, we could proceed at once to the case of M_{r-1}, which will need separate consideration.

Now M_0 is a C^1-submanifold [7], because G acts in a locally orthogonal manner. Since it has been assumed that $0 < r - 1$, it is known [1] that

$$\dim M_0 \leq n - 3,$$

and therefore by Lemma 2 there is a triangulation of M such that no 2-cell of the triangulation meets M_0. By the standard deformation theorems, $f(\sigma)$ may be deformed to be in the two-skeleton of the triangulation of M, and thus to a position not meeting M_0. We continue to denote the deformed $f(\sigma)$ by $f(\sigma)$. The deformation can and will be assumed to take place without altering α by more than a preassigned amount, and this deformation of α does not affect the proof.

Next assume $i = 1$, $1 < r - 1$. Then $f(\sigma)$ may intersect M_1. Here we proceed step by step on the sets M_{1j}; it is sufficient to consider a finite number of indices j because it is known that there are at most a finite number of conjugacy classes of isotropy groups [6]. Let t be the largest value of j which needs to be considered, so that $M_{1t} \cup M_0$ is closed. The set M_{1t} is a C^1-submanifold and contains a finite C^1-complex K_t such that

$$f(\sigma) \cap M_{1t} \subset K_t \subset M_{1t}.$$

By the assumption $1 < r - 1$, it follows that

$$\dim K_t \leq \dim M_{1t} \leq \dim M_1 \leq n - 3.$$

By Lemma 2 we may find a triangulation of M such that no 2-cell in the triangulation of M meets K_t. Then $f(\sigma)$ may be deformed into the 2-skeleton by a deformation so slight that it does not create an intersection with M_0, and now $f(\sigma)$ does not meet $M_0 \cup M_{1t}$. By continuing to M_{1t-1}, M_{1t-2}, $\cdots$, we obtain in a finite number of steps a deformed $f(\sigma)$ such that

$$f(\sigma) \cap [M_0 \cup M_1] = \emptyset .$$

We continue to M_2 and analyze M_{2j} step-by-step on j, and so on; in this way we obtain, after a finite number of steps, a deformed $f(\sigma)$ such that

$$f(\sigma) \cap [M_0 \cup M_1 \cup \cdots \cup M_{r-2}] = \emptyset .$$

Thus $f(\sigma)$ now satisfies the condition $f(\sigma) \cap B \subset M_{r-1}$.

We next wish to examine the set $A \subset M_{r-1}$, where $m(x | M_{r-1})$ is discontinuous. Take $p \in A$, and let K be a slice at p; assume $m(p) = a$. Since $p \in A$, there must be points y,

$$y \in K \cap M_{(r-1)j} \qquad (j < a).$$

Let

$$\beta = K \cap F(G_p) .$$

Then β is a closed rectilinear cell in $M_{(r-1)a}$, and $\dim \beta \leq \dim K - 2$. There are a finite number of points $y_1, \cdots, y_s$ in K such that if and only if $y \in K \cap B$ and $m(y) < a$, then

$$G_y = G_{y_i} \text{ for some } i \ (1 \leq i \leq s) .$$

Let

$$\beta_i = K \cap F(G_{y_i}) ,$$

so that β_i is a closed rectilinear cell not identical with β.

The points of A in K are formed by the intersections of pairs of the cells $\beta, \beta_1, \cdots, \beta_s$. The set $A \cap K$ is therefore the union of certain rectilinear cells of dimension at most $(\dim K - 3)$. This proves the following:

LEMMA 3. *Any compact part of* A *is contained in the union of a finite set of* C^1-*complexes of dimension at most* $n - 3$.

The Lemma, together with the results on deforming $f(\sigma)$ already obtained, enables us to conclude that $f(\sigma)$ may be deformed so that

$$f(\sigma) \cap B \subset M_{r-1} - A .$$

Among the components of the sets $M_{(r-1)j}$ there are some which have dimension at most $n - 3$. By the procedures already outlined, we may deform $f(\sigma)$ so that it does not interesect any such components. As a matter of fact, it is not necessary to proceed in any particular order to do this, since $f(\sigma)$ does not touch any points where two such components might come together. Therefore we may now assume that

$$f(\sigma) \cap B \subset C .$$

Now let $f^*(\sigma) = Tf(\sigma)$, so that

$$f^*(\sigma) \cap B^* = f^*(\sigma) \cap C^*,$$

and to complete the proof of the theorem we must deform $f^*(\sigma)$ into $U^* \cup D^*$.

Let $b^* \in f^*(\sigma) \cap C^*$, and let β^* be the $(n - r)$-cell whose existence is asserted in Lemma 1. We see that b^* and points of C^* in a neighborhood of b^* may be deformed into the interior of β^*. This may be done so as not to introduce any new points into $f^*(\sigma) \cap C^*$. Since $f^*(\sigma) \cap C^*$ is compact, we obtain, in a finite number of steps, a deformation of $f^*(\sigma)$ which does not touch B^*. This completes the proof of the theorem.

For the proof of the corollary we proceed as follows: Take α^* in U^* with α a covering path bounding $f(\sigma)$. We may assume

$$f(\sigma) \cap B \subset C.$$

If $\dim D \leq n - 3$ we may deform away from it, as for the part of B having dimension $\leq n - 3$. After this,

$$f^*(\sigma) \subset U^* \cup C^*,$$

and we may deform away from C^* as before. This proves the corollary.

4. PROOF OF THEOREM 2

LEMMA 4. *Under the hypothesis of Theorem* 1, $U^* \cup E^*$ *is simply connected.*

Let α^* be in $U^* \cup E^*$, and let α be a covering path in $U \cup E$. Let $f(\sigma)$ be a singular 2-cell with boundary α. By the proof of Theorem 1, it may be assumed that

$$f(\sigma) \cap B \subset C.$$

Let Q be the set of points of D where $m(x|D)$ has a discontinuity. Take $d \in Q$; let K be a slice at d, and let

$$\beta = F(G_d) \cap K,$$

so that β is a rectilinear cell of dimension at most $(\dim K - 2)$. There is a finite set of points $y_1, y_2, \cdots, y_s$ in K such that $m(y_i) < m(d)$, and if $y \in D \cap K$ and $m(y) < m(d)$, then

$$G_y = G_{y_i} \text{ for some } i \ (1 \leq i \leq s).$$

Let

$$\beta_i = F(G_{y_i}) \cap K.$$

Then β_i is a rectilinear cell, and points of $Q \cap K$ are made up by intersections of pairs of the cells $\beta, \beta_1, \cdots, \beta_r$, and therefore $f(\sigma) \cap Q$ is in a finite C^1-complex, in D, of dimension $\leq n - 3$. Therefore $f(\sigma)$ may be deformed so that

$$f(\sigma) \subset U \cup (D - Q).$$

Among the components of the sets M_{rj} $(j > k)$ there are some which have dimension at most $n - 3$. By the standard procedure, $f(\sigma)$ may be deformed so as to avoid these components. Therefore it may now be assumed that

$$f(\sigma) \subset U \cup C \cup E, \qquad f^*(\sigma) \subset U^* \cup C^* \cup E^*.$$

But as at the conclusion of the proof of Theorem 1, we may deform $f^*(\sigma)$ away from C^* so that

$$f^*(\sigma) \subset U^* \cup E^*.$$

This proves the Lemma.

LEMMA 5. $U^* \cup E^*$ *is a manifold.*

The set $U^* \cup E^*$ is connected, so that to be sure it is a manifold we need only show that it is locally euclidean at each point. It is known to be locally euclidean at points of U^*, and it remains only to examine points of E^*.

Take $p^* \in E^*$ and $p \in E$, so that $T(p) = p^*$, and let K be a slice at p. Then

$$L_1 = K \cap F(G_p)$$

is a cell of dimension $(n - r - 2)$, and we may assume that

$$K = L_1 \times L_2,$$

where L_2 is a 2-cell orthogonal to L_1. For $g \in G_p$ and $k = (l_1, l_2)$ in K, we have

$$g(k) = g(l_1, l_2) = (l_1, gl_2).$$

Thus the action of G_p on K is determined by its action on L_2. If N is the subgroup of G_p leaving all of K fixed, then N is a normal subgroup of G_p, and G_p/N acts effectively on L_2. Since G_p/N is finite and since $p \in E$, G_p/N is a cyclic group equivalent to a cyclic group of rotations. Hence L_2^* is a 2-cell. But L_1^* is an $(n - r - 2)$-cell, and

$$K^* = L_1^* \times L_2^*$$

is an $(n - r)$-cell which is a neighborhood of p^*. This completes the proof of the Lemma.

The proof of Theorem 2 may now be completed: The manifold $U^* \cup E^*$ is simply connected and is therefore orientable. But U^* is a submanifold, and hence U^* is orientable.

COROLLARY 1. *Let a compact connected Lie group act differentiably on* $M = R^n$ *or* S^n. *Then no closed path in* U^* *can reverse the orientation of an orbit in* U; *that is, the* r*-dimensional homology of an orbit in* U *forms a constant sheaf over* U^*.

The orbits in U form a local product, and therefore each of them is orientable. However an even stronger result is true, as follows:

COROLLARY 2. *Let a compact connected Lie group* G *act differentiably on* $M = R^n$ *or* S^n. *Then every orbit of highest dimension is orientable.*

COROLLARY 3. *Let a compact connected Lie group act differentiably on* $M = R^n$ *or* S^n. *If* $n - r$ *is odd, there can be no isolated orbits in* D.

In order to prove Corollary 1, let α^* be a closed path in U^*, and let α be a covering closed path in U with end points at p. If a motion around α reversed the orientation of G(p), it would also reverse the orientation of a slice at p. This would imply that U^* is nonorientable, and this is impossible. This proves Corollary 1.

To prove Corollary 2, let G(p) be an orbit of highest dimension, and let K be a slice at p. If G(p) were nonorientable, there would be an element $g \in G_p$ which reverses the local orientation of G(p). Such an element would reverse the orientation of a slice at p. There is an arc in G(p) which joins p to g(p), and there is an arc in K - D joining g(p) to p. The first arc reverses the orientation of K, and the second preserves it because K is orientable. Hence the union of these two arcs is a closed path in U which reverses the local orientation of the slice K. But this is impossible, and this contradiction proves Corollary 2.

To prove Corollary 3, let p be a point of D and let K be a slice at p. Let N be the subgroup of G_p leaving all of K fixed, and consider the finite group G_p/N which operates on K. Since D is isolated, the group G_p/N operates freely on K - p, and each of its elements preserves the orientation of K. But if a is an element of G_p/N, a is cyclic, and if it operates freely, $n - r$ must be even, so that the spheres in K - p have odd dimension.

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