THE DISTRIBUTION OF THE a-POINTS OF CERTAIN MEROMORPHIC FUNCTIONS

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INTRODUCTION

With reference to the value-distribution theory of Rolf Nevalinna [13], modern research is confronted with a problem formulated by Oswald Teichmüller [17] as follows:

If W *is a simply connected Riemann surface over a* w-*sphere, then it is known that* W *can be mapped one-to-one and conformally onto either the disk* $|\zeta| < 1$, *the plane* $\zeta \neq \infty$, *or the complete* ζ-*sphere, so that* W *becomes a single-valued function of* ζ, $W = W(\zeta)$. *The value-distribution of this function is to be investigated.*

We are as yet far from a general solution of the main problem of geometric value-distribution. However, there are two special classes of surfaces for which the solution is attainable. The two classes are formed of those surfaces whose graphs (in the sense of Speiser [15], Nevanlinna [13] and Elfving [2]) are formed of a finite number of simply periodic ends or doubly periodic ends. For these classes we have given explicit formulas for determination of order, defect and branch-indices.

In the present paper we obtain further geometric properties of the distribution of a-points of a meromorphic function in one of the two classes described.

1. THE GRAPHS

We first consider some elementary properties of the graphs. We restrict attention to a simply connected Riemann surface over the w-sphere, with all branch or boundary points lying over a finite number of base points $a_1, \cdots, a_p$. We connect these points by a simply closed path L which separates the sphere into a positively-circulated region J (inner region) and a negatively-circulated region A (outer region). The points above L on the Riemann surface form axes which together provide a cell decomposition of the surface; each cell is a half-sheet lying over J or over A. We now choose one point (node) in each such half-sheet. If two half-sheets have one or more boundary axes in common, then we join the corresponding nodes by an arc. The result is a graph in the surface, dual to the graph formed by the axes of the cell decomposition. Each such graph has a finite or infinite set of inner nodes (indicated by small circles) and outer nodes (indicated by small crosses), as shown in Figure 1.

To a branch point of order n, there corresponds a polygon with 2n vertices in our graph. If $n = 1$, the basic point is covered simply; and $n = \infty$ corresponds to a logarithmic branch point in the w-sphere and to a "logarithmic elementary region" in the surface.

Following E. Ullrich [19] we now choose a rational function R(t) and form the function $w = R(e^z)$. The inverse function has a Riemann surface with two logarithmic branch points, over the points R(0) and R(∞) of the w-sphere. [If $R(t) = (t + t^{-1})/2$, we obtain the function $w = \cos z$.] It is evident that each graph for a surface of this

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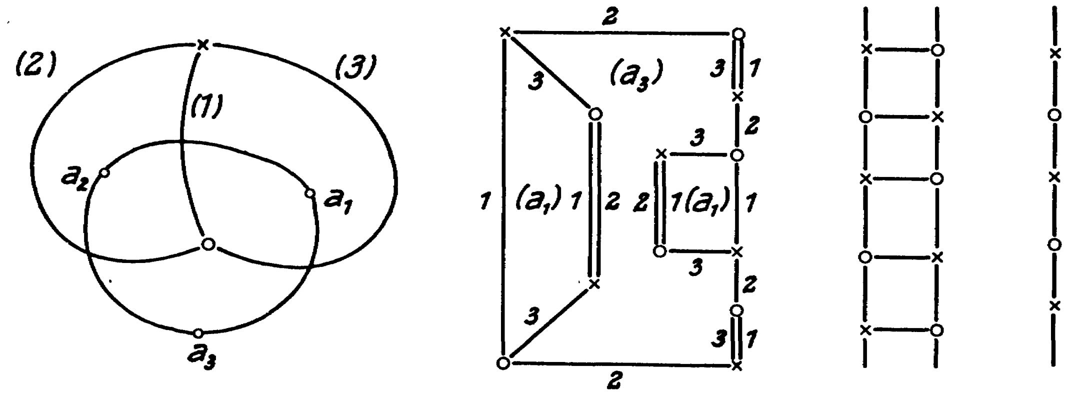

Figure 1

type is simply periodic. One half of such a graph we call a periodic end. If a surface has a graph formed of several periodic ends emanating from a common nucleus (see Figure 2), then, by definition, the surface belongs to one of the two classes mentioned in the Introduction. Between two adjacent periodic ends there is a logarithmic elementary region. We can determine the value-distribution of the corresponding function (meromorphic for $|Z| < \infty$) from the graph alone.

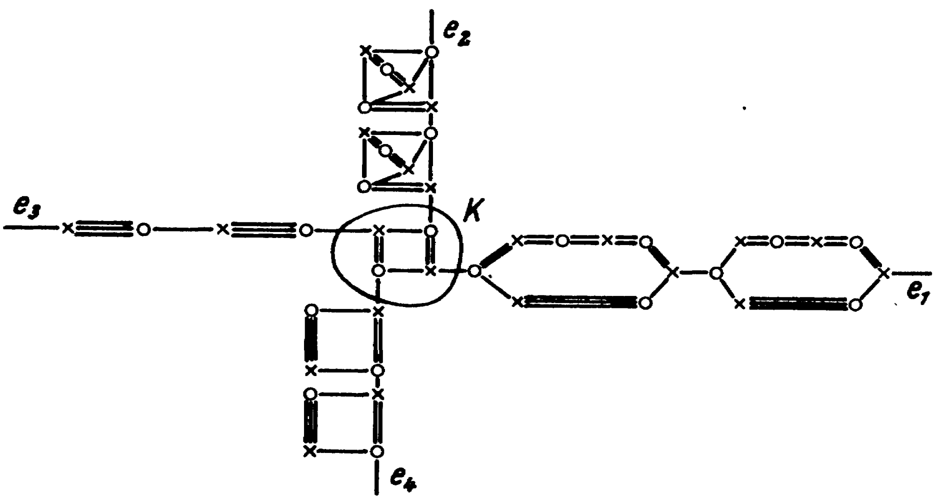

Figure 2

In addition to the simply periodic graphs obtained from the functions $R(e^z)$, we are interested in doubly periodic graphs which are obtained in similar manner with the aid of doubly periodic functions (see Figure 3). One half of such a doubly periodic graph is called a doubly periodic end. From the two graphs shown in Figure 3, we obtain the two ends shown in Figure 4. Doubly periodic ends were first mentioned by E. Ullrich [18] and O. Teichmüller [17].

A graph may consist of several doubly periodic ends. Or it may consist of several simply periodic and doubly periodic ends emanating from a common nucleus; in this case, it is clear that between two adjacent ends there must be a logarithmic elementary region. For surfaces whose graphs are of either of these two types, the value-distribution can also be calculated from the graph alone.

In an earlier investigation [9], it was shown that the surfaces capable of representation by doubly periodic ends have no deficiencies. This is also valid for the

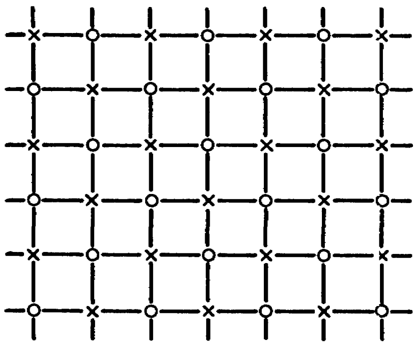

Figure 3a

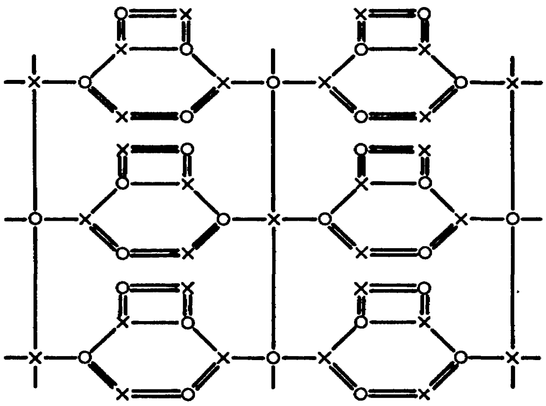

Figure 3b

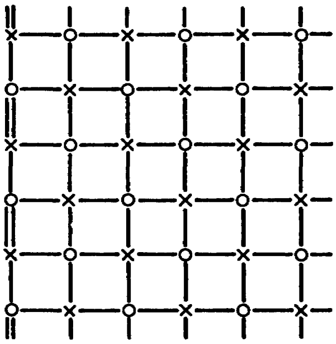

Figure 4a

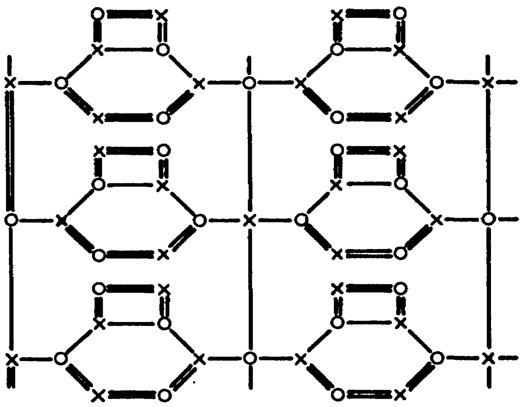

Figure 4b

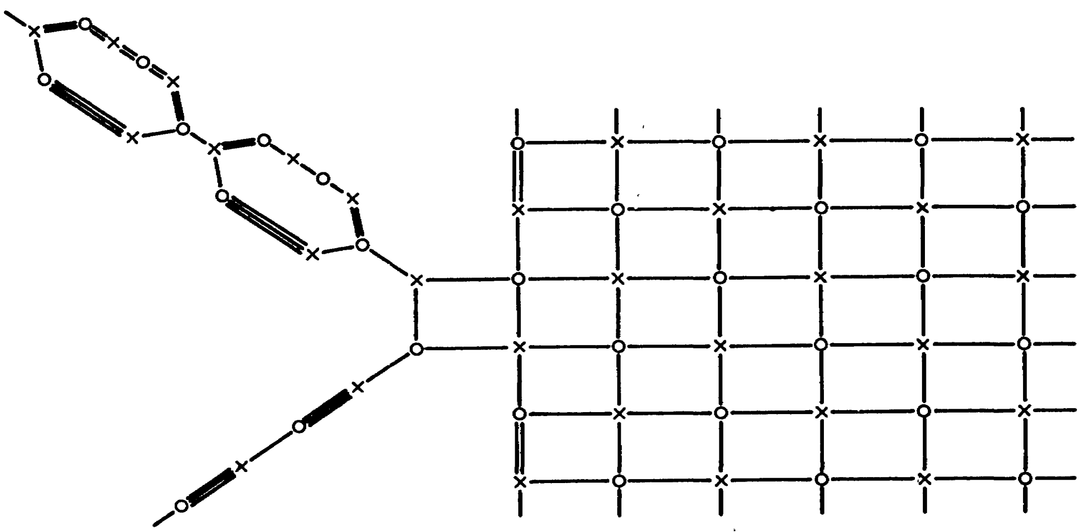

Figure 5

mixed class. It is thus clear that one doubly periodic end suffices to remove the deficiencies caused by simply periodic ends (Figure 5).

2. APPLICATION OF FUNCTION THEORY

For the study of our special class of surfaces, quasiconformal mappings are especially suitable. By a quasiconformal mapping we shall mean a topological mapping $w = T(Z)$ of a plane domain $\mathcal{D}$ of the z-plane $(z = x + iy)$ onto a domain Δ of the w-plane $(w = u + iv)$ having continuous partial derivatives with respect to x and y except at certain isolated points. Thus $|dw/dz|$ is in general dependent on $\arg dz = \phi$. The dilatation quotient D is defined as the ratio of major to minor axis of the image ellipse of an infinitesimal circle;

(2.1)
$$D = D_{z/w} = \max_{\phi} \left|\frac{dw}{dz}\right| \div \min_{\phi} \left|\frac{dw}{dz}\right| = D_{w/z}.$$

In general, some restriction is imposed on the values of D in $\mathcal{D}$. One has necessarily $D \geq 1$, and $D = 1$ at a point implies conformality at that point. The following relation is useful for computation:

(2.2)
$$D(z) = |K| + \sqrt{K^2 - 1}, \qquad K = \frac{1}{2}\frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{u_x v_y - u_y v_x}.$$

Let $w(z)$ be a quasiconformal mapping of the z-plane onto the w-plane such that $w(\infty) = \infty$. If $w(z)$ deviates only slightly from conformality—for example, if the integral

(2.3)
$$\iint_{|z| > T_0} (D_{z/w} - 1)\frac{dx\,dy}{|z|^2}$$

converges—then one has the relation of Teichmüller and Belinsky [16], [1]:

(2.4)
$$w = \Gamma z\{1 + \varepsilon(z)\} \qquad (\Gamma = \text{constant}, \quad \lim_{|z| \to \infty} \varepsilon(z) = 0).$$

The distortion formula contained in this theorem was proved by Teichmüller [16] and Wittich [22]; the former established the formula with the aid of the modulus theorem; the latter established it by applying the differential equation of Ahlfors. The distortion formula states that a circle with large radius is mapped on a curve differing slightly from a circle. The torsion formula contained in the theorem was recently proved by Belinsky [1]. He made use of various auxiliary theorems, some of which are due to Grötzsch and Lavrentieff. According to Belinsky's result, convergence of the integral (2.3) implies that arg w varies only slightly as $|z| \to \infty$, with $\arg z = $ constant. (Recently, R. Nevanlinna [14] has also proved the torsion formula, but under stronger hypotheses.)

In order to determine properties of the value-distribution, we decompose the Riemann surface into a finite number of parts. We then uniformize the parts by corresponding functions and piece together the images with the aid of quasiconformal mappings satisfying (2.4). The result is a composite quasiconformal z-image of the Riemann surface which, by known theorems, is also quasiconformally related, in the sense of (2.4), to the conformal image of the Riemann surface in a ζ-plane. Accordingly, we are able to determine the desired properties in the quasiconformal z-image instead of the conformal ζ-image. This possibility will frequently be exploited in the investigations to be described.

The uniformization of a surface with simply periodic ends is carried out as follows.

First we place a closed curve C about the nucleus, which consists of a finite number of inner and outer nodes. The portion enclosed by C corresponds to a compact subregion on the Riemann surface. As such, it is of no significance for the asymptotic relations of value-distribution and can be disregarded in the following.

Our second step is to consider the logarithmic branch point a_n. If we enclose a_n in a sufficiently small circle of radius t, the branch element $|w - a_n| < t$ can be mapped conformally onto a half-plane by means of a logarithmic function. In this manner, p half-planes are obtained as images of the p branch elements.

The third and last step concerns the strip-neighborhood of the simply periodic ends. After uniformization of the nuclear region and the logarithmic elementary regions, there remain p half-strips corresponding to these ends. These regions on the Riemann surface are uniformized with the aid of the inverse functions of $w_n = R_n(\exp \zeta_n)$ $(n = 1, 2, \cdots, p)$, where $R_n(t)$ is the rational function associated with the nth periodic end. The p half-strips must be joined to the p half-planes along corresponding boundaries, in accordance with the corresponding decomposition of the Riemann surface; this is achieved with the aid of quasiconformal mappings, as shown schematically in Figure 6, where $e_1', e_1'', \cdots, e_p', e_p''$ are the halves of periodic ends. We now arrange these new half-planes in the proper manner alternately over the negative and positive real axis in a Z-plane $(z = Re^{i\phi})$. It remains to identify the free edges. To achieve this, we make use of the spiral mapping

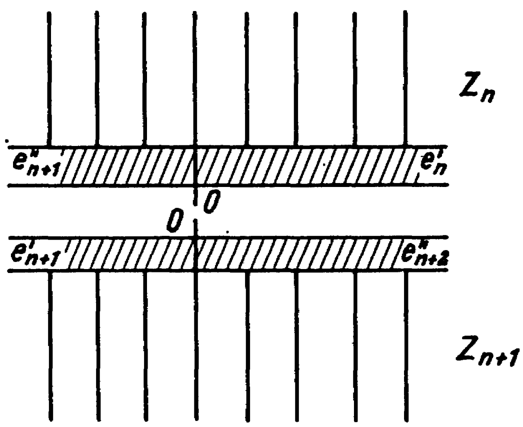

Figure 6

$$Z = z^{\alpha + i\beta} \tag{2.5}$$

by which the Riemann surface is mapped from the Z-plane to the z-plane. In the mapping

$$\alpha = \frac{p}{2}, \quad \beta = -\frac{\log A}{2\pi}, \tag{2.6}$$

where

$$A = \frac{\omega_1 \omega_2 \cdots \omega_p}{\omega_1' \omega_2' \cdots \omega_p'}.$$

Here ω_n denotes the number of inner nodes (which equals the number of outer nodes) on the right-hand boundary of one period of the end, and ω_n' is the corresponding number for the left-hand boundary. Right and left are interpreted relative to the nucleus. By virtue of (2.6) and spiral mapping,

$$Z = z^{p\left(1 - i\frac{2}{p}\frac{\log A}{2\pi}\right)/2}. \tag{2.7}$$

Thus our Riemann surface W has been uniformized quasiconformally and, by (2.4), there is an asymptotic equation

$$\zeta = \gamma z\{1 + \varepsilon(z)\}. \tag{2.8}$$

Thus it is permissible, as stated above, to observe first the properties of the value-distribution in the z-plane and then to transform these to a conformally related ζ-plane by means of (2.8).

A simple example of doubly periodic ends is shown in Figure 4a. Without loss of generality, we normalize the base points a_i $(i = 1, 2, 3, 4)$ of the surface so that $a_1 + a_2 + a_3 = 0$ and $a_4 = \infty$. The only logarithmic branch point is at a_1. We uniformize a circular neighborhood of this point by a logarithmic function mapping it into a Z-plane (as for the simply periodic ends above). The second step is the mapping of the remaining region of the Riemann surface onto a half Z-plane by means of the inverse of the Weienstrass $\wp$-function: that is,

$$Z = X + iY = \int_{a_4}^{w} \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}} + k. \tag{2.9}$$

Again the two Z half-planes can be joined by means of quasiconformal mappings in accordance with (2.4). If doubly periodic ends such of those in Figure 4b occur, the Weierstrass $\wp$-function is replaced by an elliptic function of the form

$$f(u) = R_1[\wp(u)] + \wp'(u)R_2[\wp(u)]. \tag{2.10}$$

For the mixed class in which the graphs have p simply periodic ends and q doubly periodic ends, the uniformization is carried out as in the previous cases. It is evident that we now have p + 2q half-planes, which must be joined alternately over the negative and positive real axes. In this case also a spiral mapping analogous to (2.5) converts the surface into a schlicht z-plane with the desired identification of the two remaining free edges.

We have now given a survey of the theory of simply and doubly periodic ends. In the following sections we investigate the problems of the maximum modulus and of the distribution of a-points for certain entire and meromorphic functions.

3. ON THE MAXIMUM MODULUS OF SOME ENTIRE FUNCTIONS

We now treat those transcendental entire functions that have Riemann surfaces representable by graphs formed of a finite number of simply periodic ends. Thus the point $w = \infty$ remains uncovered.

We select three exceedingly simple graphs S_1, S_2 and S_3 (Figure 7) and choose $-1, +1, \infty$ as base points in the w-plane. For these three cases we seek the points ζ for which

$$M(r) = w(\zeta),$$

where $M(r)$ is the maximum modulus of $w(\zeta)$ for $|\zeta| = r$.

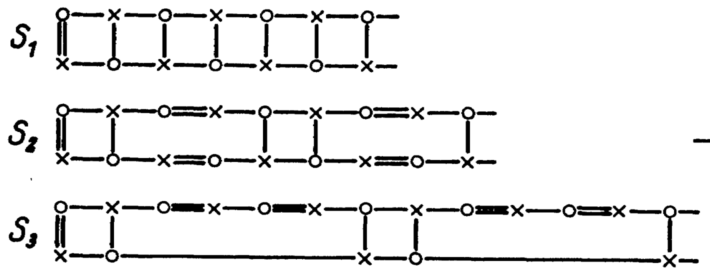

Figure 7a

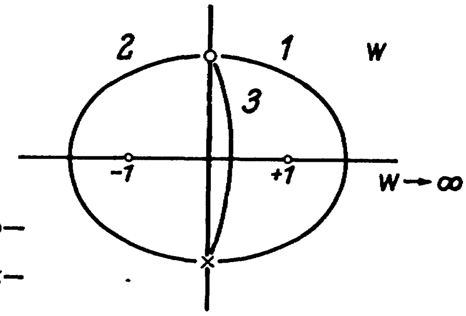

Figure 7b

For the graph S_1 we can exhibit the function explicitly; it is the function $w = \cos\sqrt{\zeta}$, which assumes its maximum modulus on the negative real axis, as can be shown by a simple calculation. The roots of the equation $\cos\sqrt{\zeta} \pm 1 = 0$ are to be found on the positive real axis.

For the graph S_2 it is no longer possible to describe the function explicitly in an elementary manner. For this reason we employ the uniformization methods of Section 2 and obtain as image in the z-plane a half-plane, as shown schematically in Figure 8a. Since in this case $\omega_1 = \omega_1'$, it follows that $\beta = 0$; our function (2.5) reduces

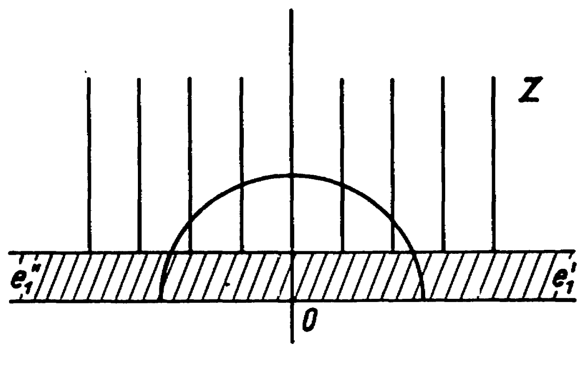

Figure 8a

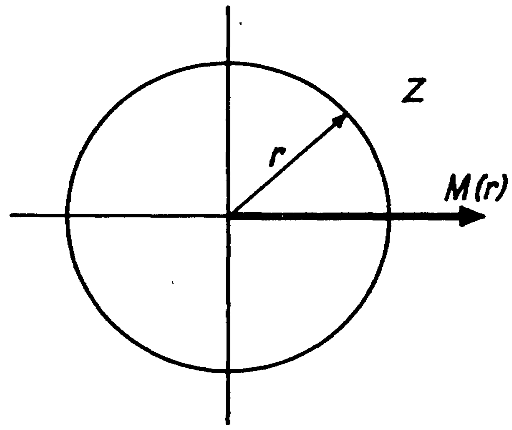

Figure 8b

to $Z = \sqrt{z}$. Clearly the maximum modulus in the uniformized z-plane is taken on a ray $\arg z =$ constant. According to (2.9), the maximum modulus for the function associated with S_2 will be found on the conformal image of a ray.

For the graph S_3 (Figure 7) the analysis is different. This graph has asymmetry with reference to the boundary nodes. The spiral mapping (2.5) now has constants $\alpha = 1/2$ and $\beta = -(2\pi)^{-1}\log 3$ (Figure 9). The ray $\arg Z = \pi/2$ in the Z-plane, on which the maximum modulus is assumed, is transformed into the logarithmic spiral

$$\log r = \frac{\pi}{\log \zeta}\,\phi - \pi. \tag{3.1}$$

In this case also we deduce from the formula (2.8) that the transcendental entire function $w = w(\zeta)$ corresponding to S_3 has its maximum modulus on a curve, only slightly deviating from the logarithmic spiral (3.1).

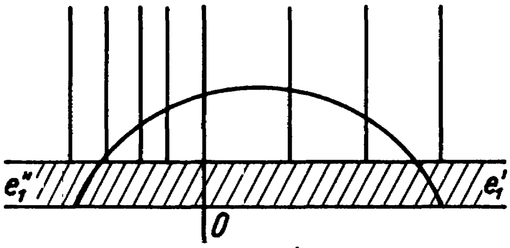

Figure 9a

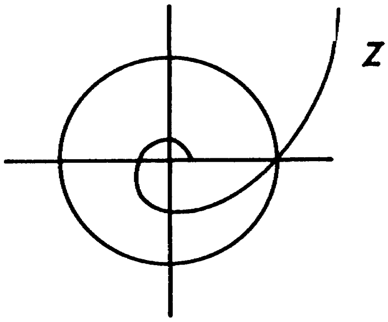

Figure 9b

4. DISTRIBUTION OF THE a-POINTS

Again those functions are of interest that have graphs represented by one of the following:

1. a finite number of simply periodic ends;

2. a finite number of doubly periodic ends;

3. a combination of p $(p < \infty)$ simply periodic ends and q $(q < \infty)$ doubly periodic ends.

Our task is to determine for such functions the distribution or arrangement of the a-points in the ζ-plane. Two cases must be distinguished, those in which $\beta = 0$ and those in which $\beta \neq 0$.

Applying known methods, we first uniformize the Riemann surface piecewise and, after the desired identification, we obtain a surface over the Z-plane consisting of parallel periodic strips of width $2\pi i$ (this concept has no meaning for quasiconformally mapped domains, but that is of no import for our asymptotic investigations). Over this Z-plane, the a-points lie on straight lines $Y = h$. We must now map these straight lines onto the z-plane or the ζ-plane.

We consider first the case $\beta = 0$. This leads to the transformation $z = Z^{1/\alpha}$ $(\alpha \geq 1/2)$, or

$$(4.1)\quad \begin{cases} x = X^{\alpha^{-1}} - \binom{\alpha^{-1}}{2} X^{\alpha^{-1}-2} Y^2 + \binom{\alpha^{-1}}{4} X^{\alpha^{-1}-4} Y^4 - \cdots, \\ y = \binom{\alpha^{-1}}{1} X^{\alpha^{-1}-1} Y - \binom{\alpha^{-1}}{3} X^{\alpha^{-1}-3} Y^3 + \cdots. \end{cases}$$

We shall consider the following special cases:

$$\alpha = 1/2: \quad x = X^2 - Y^2, \quad y = 2Xh;$$

$$\alpha = 1: \quad x = X, \quad y = h;$$

$$\alpha = 3/2: \quad x = X^{2/3} + \frac{1}{9} h^2 X^{-4/3} + \cdots, \quad y = \frac{2}{3} h X^{-1/3} + \cdots.$$

If we use polar coordinates: $Z = Re^{i\theta}$, $z = re^{i\phi}$, with

$$R = \sqrt{X^2 + h^2}, \qquad \theta = \tan^{-1}\frac{h}{x} + n\pi,$$

we obtain the representation in the z-plane:

$$\text{(4.1')} \qquad \begin{aligned} \alpha \log r &= \log X + O(t^2), \\ \alpha\phi &= \frac{h}{X} + n\pi + O(t^2), \end{aligned}$$

with $t = h/X$.

By the distortion theorem, the a-points (zeros of $w(\zeta) - a$) are to be found in the ζ-plane in narrow strips, which can be determined by means of transformations (4.1) and (4.1').

If $\beta \neq 0$, the straight lines $Y = h$ have the following representation:

$$\text{(4.2)} \qquad \begin{cases} (\alpha^2 + \beta^2) \log r = \alpha \log X + \beta\frac{h}{X} + \beta n\pi + O(t^2), \\ (\alpha^2 + \beta^2) = -\beta \log X + \alpha\frac{h}{x} + \alpha n\pi + O(t^2), \end{cases}$$

or

$$\text{(4.2')} \qquad \beta \log r + \alpha\phi = \frac{h}{X} + n\pi + O(t^2),$$

from which it follows that, for $|X| \to \infty$,

$$\text{(4.2'')} \qquad \beta \log r + \alpha\phi = n\pi.$$

Again, in accordance with the distortion theorem (2.4) or (2.8), the a-points are to be found in narrow strips with spiral boundaries, called briefly spiral strips.

5. EXAMPLES

Example 1. We take the graph S_2 of Figure 7a. Here $\alpha = 1/2$ and $\beta = 0$. In the z-plane, the a-points lie on parabolas, as shown in Figure 10; in the figure, a cross denotes an a-point for which a lies in the neighborhood of the logarithmic branch point, and a small circle denotes an a-point for which a lies near $w = +1$. The transformation to the ζ-plane is obtained here, as in the cases to follow, from (2.8).

Example 2. We consider the graph S_3. Here $\alpha = 1/2$, $\beta = -(2\pi)^{-1}\log 3$. A neighborhood of the end is mapped onto a spiral strip in the z-plane (Figure 11). The boundary spirals of the strip are asymptotic to the logarithmic spiral

$$\log r = (\log 3)^{-1}\phi.$$

In the figure, a cross denotes an a-point for which a lies near the logarithmic branch point, and a circle denotes an a-point for which a is near $w = +1$.

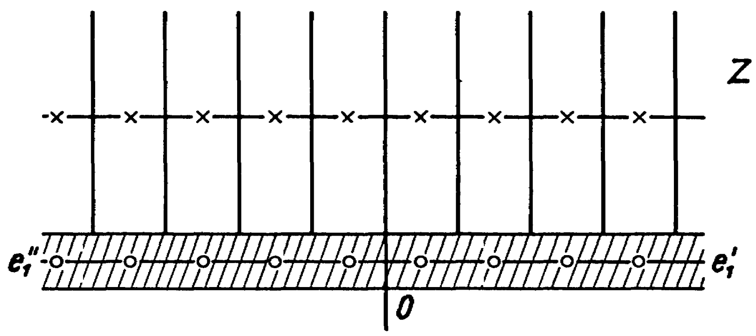

Figure 10a

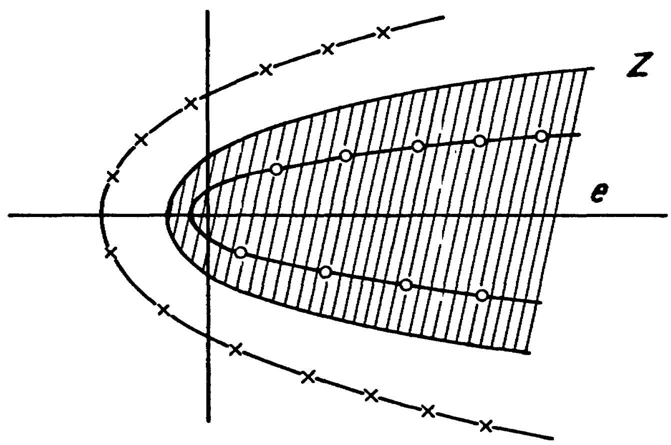

Figure 10b

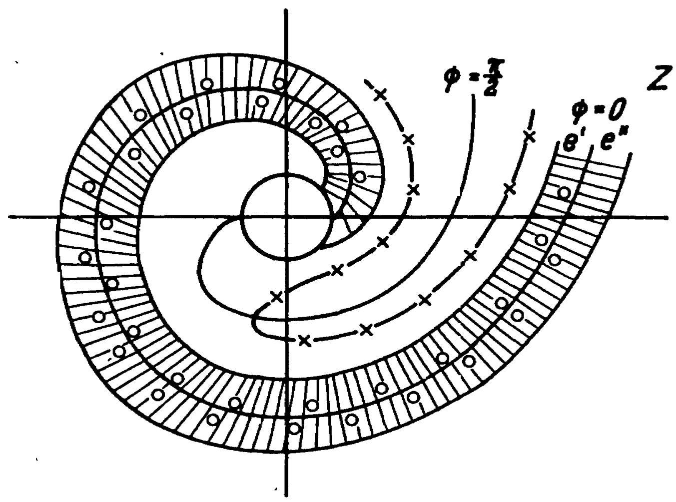

Figure 11

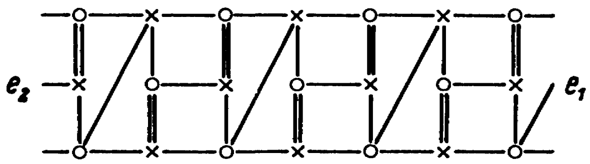

Figure 12a

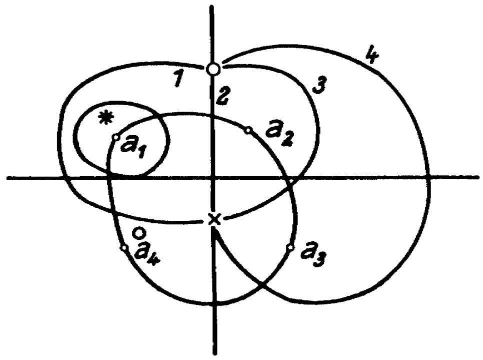

Figure 12b

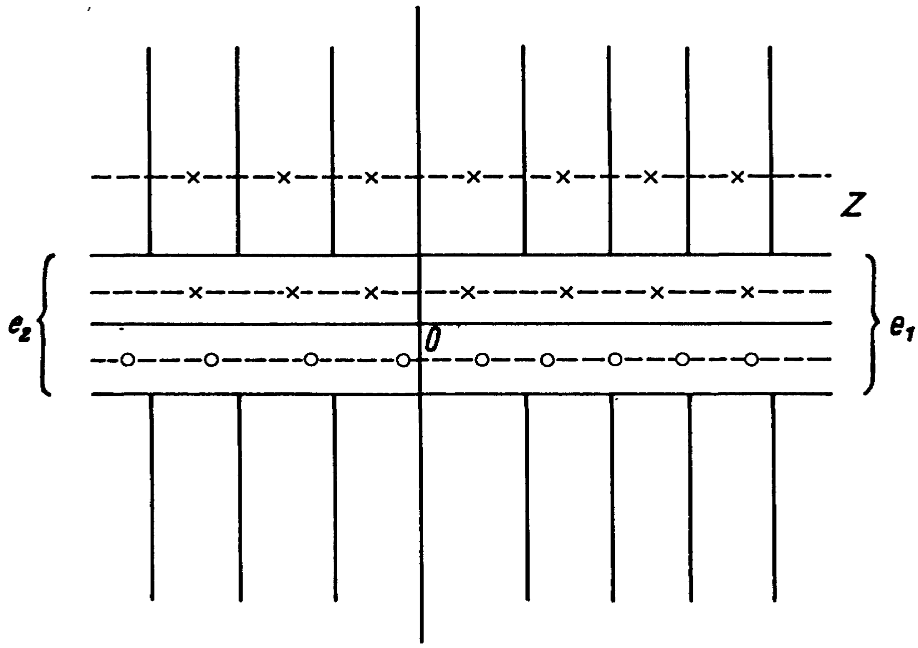

Figure 12c

Example 3. Let us consider the graph of Figure 12a with two simply periodic ends. Here we have $\alpha = 1$, $\beta = 0$. The two ends are mapped by (4.1) onto a parallel strip of finite width, as shown in Figure 12. A cross denotes an a-point for which a is near the logarithmic branch point. There are also algebraic branch points over a_1, so that additional a-points are located in the parallel strip. The circles denote a-points for which a is near a_4 (where there is an logarithmic branch point).

Example 4. The surface has a graph as shown in Figure 13a; here $\alpha = 1$, $\beta = -(2\pi)^{-1}\log 2$. Both ends are mapped onto narrow spiral strips in the z-plane. The discussion is similar to that of Example 2 (Figure 11). The cross denotes an a-point for which a is close to a_1.

Example 5. A final example of simply periodic ends is indicated in Figures 14a and 14b. In Figure 14a', the end e_j approaches a region in the z-plane containing the ray $\arg z = 2\pi j/3$ ($j = 1, 2, 3$). The boundary curves approach the ray asymptotically. As $|z| \to \infty$, the arguments of the a-points approach $2\pi j/3$. The cross denotes an a-point for which a is near the logarithmic branch point at a_1. For the graph of Figure 14b, the images of the rays $\theta = n\pi$ are determined by (4.2"). The ends e_1, e_2, e_3 in the z-plane contain the corresponding spirals. The boundary curves of the regions approach the spirals (4.2") asymptotically.

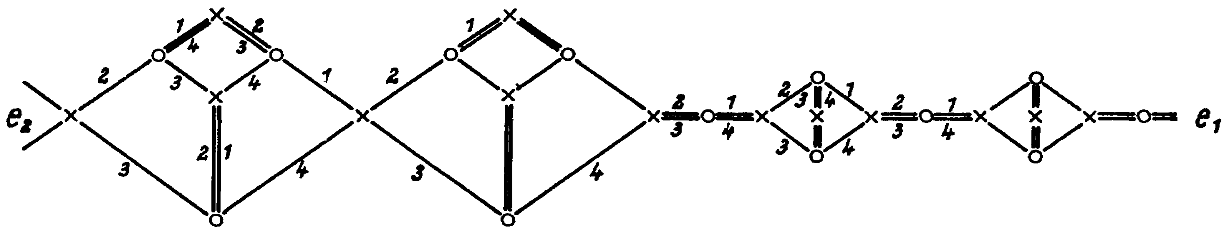

Figure 13a

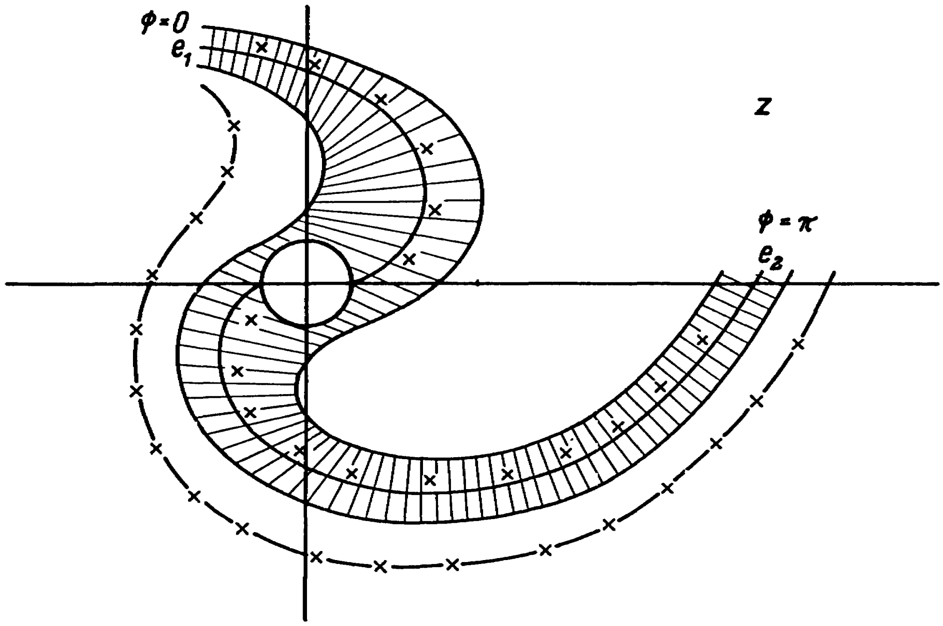

Figure 13b

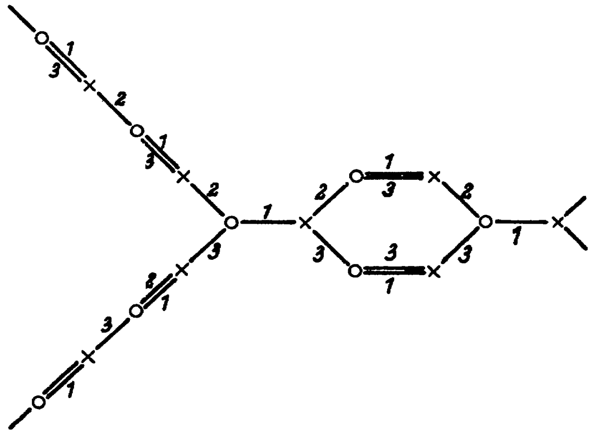

Figure 14a

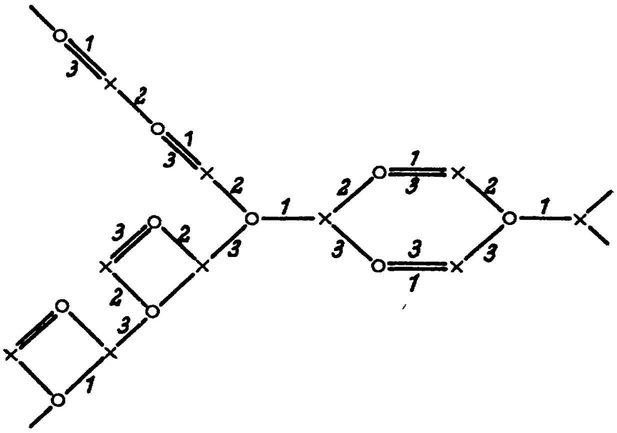

Figure 14b

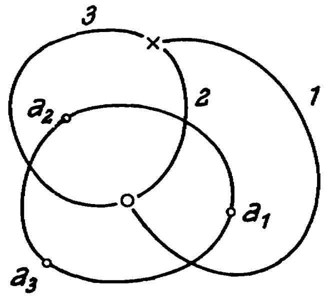

Figure 14c

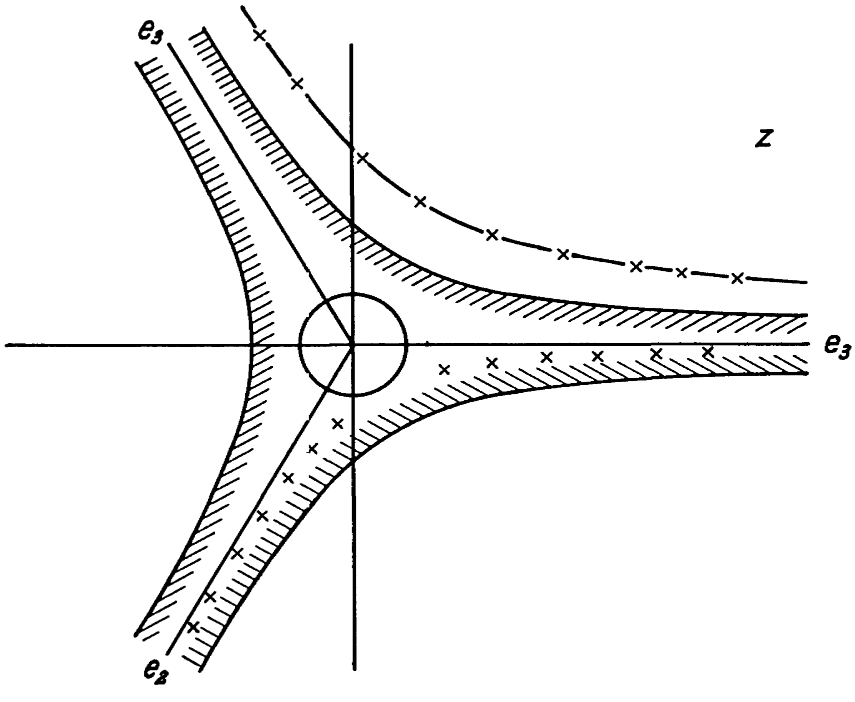

Figure 14a'

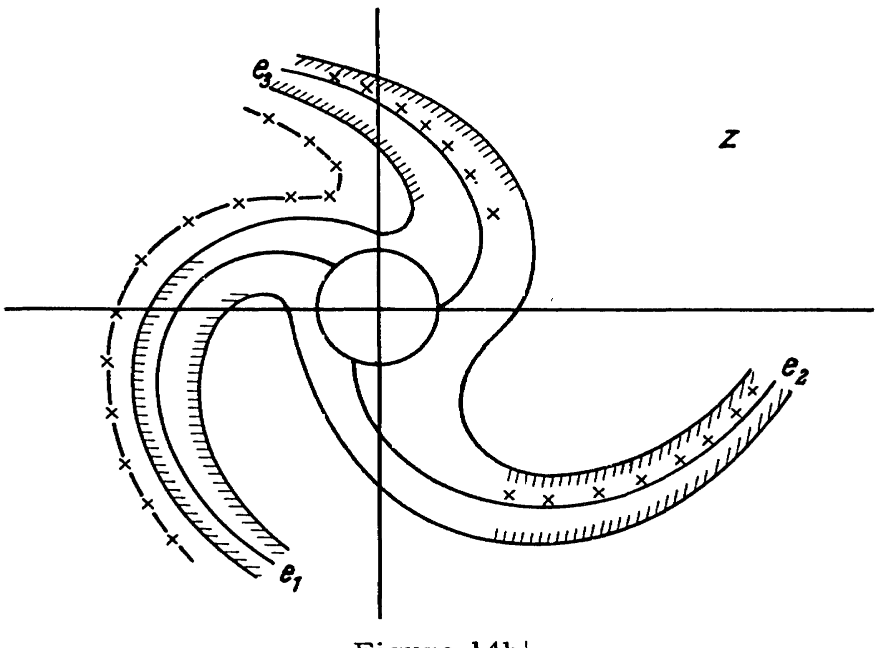

Figure 14b'

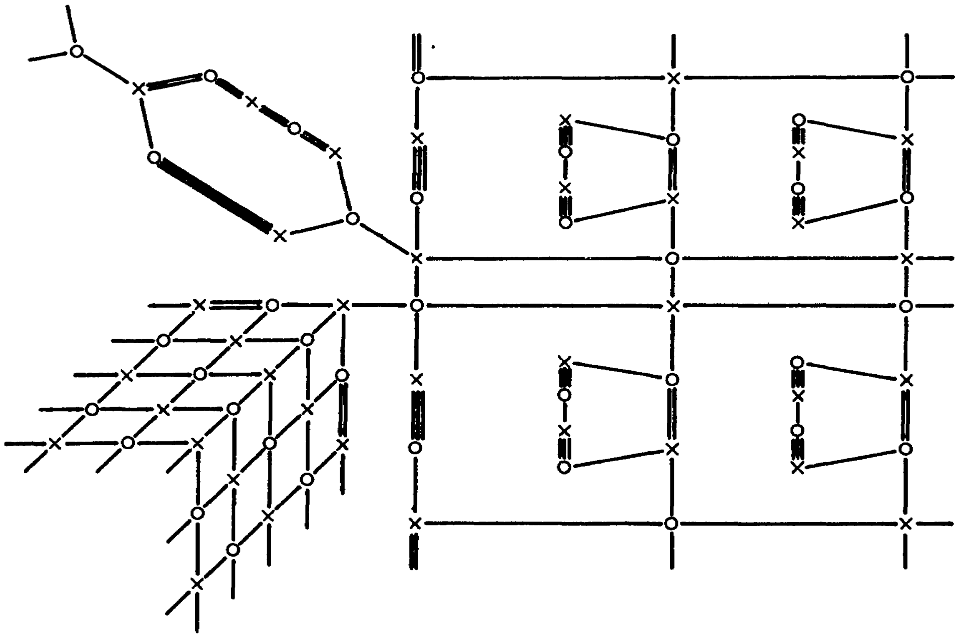

Figure 15

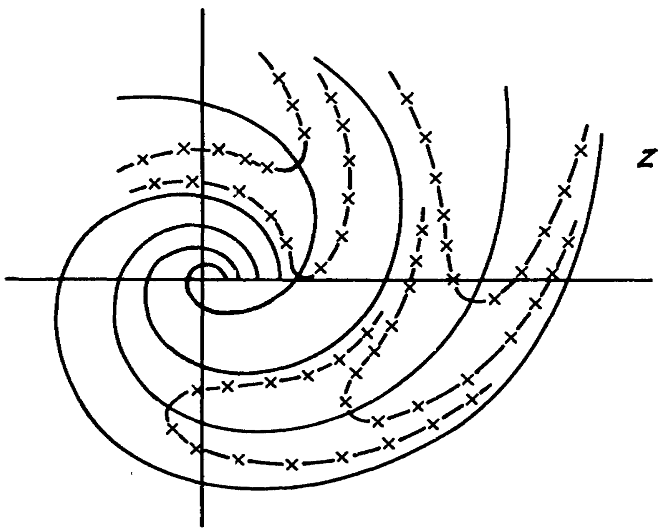

Figure 16

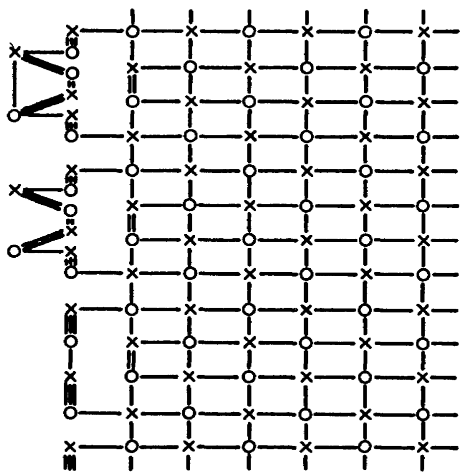

Figure 17

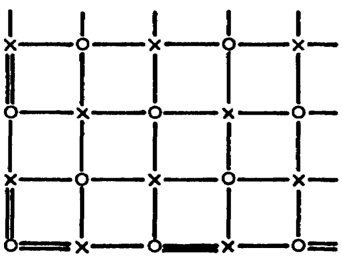

Figure 18a

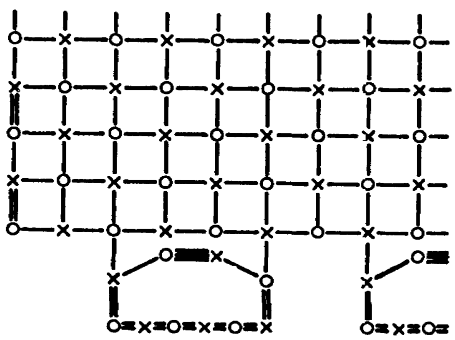

Figure 18b

Example 6. We study the distribution of a-points for the graph shown in Figure 15. The asymmetry is due solely to the simply periodic ends. The doubly periodic ends are almost exclusively responsible for the value-distribution, in particular, for the counting function n(r, a), so that we can restrict our attention to a-points arising from these ends. Because of the double periodicity, a typical a-point distribution in the Z-plane lies over an infinite number of straight lines $Y = h_i$; thus, from our previous analysis, it is apparent that the distribution in the z-plane is along two double families of spiral curves. The result is shown schematically in Figure 16.

Example 7. If we have only one doubly periodic end, a distribution following a spiral can be achieved only under the condition that the boundary be asymmetric, as illustrated in Figure 17. The spirals in the z-plane are similar to those in Figure 16.

Example 8. If we consider only a half of a double periodic end (Figure 18), we speak of a quarter-end. The quarter-ends are highly interesting from the point of view of value-distribution. As in Example 7, the boundaries can be asymmetric, so

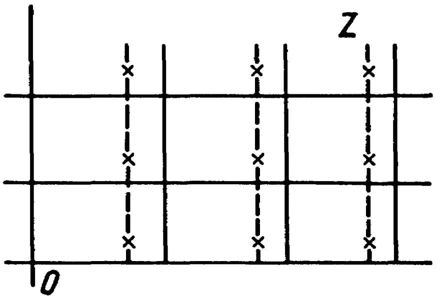

Figure 19a

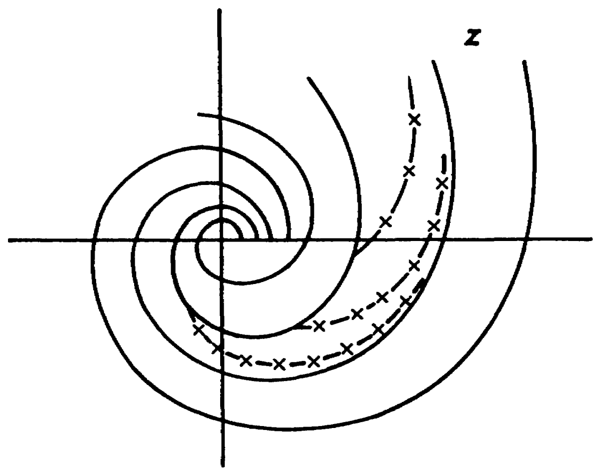

Figure 19b

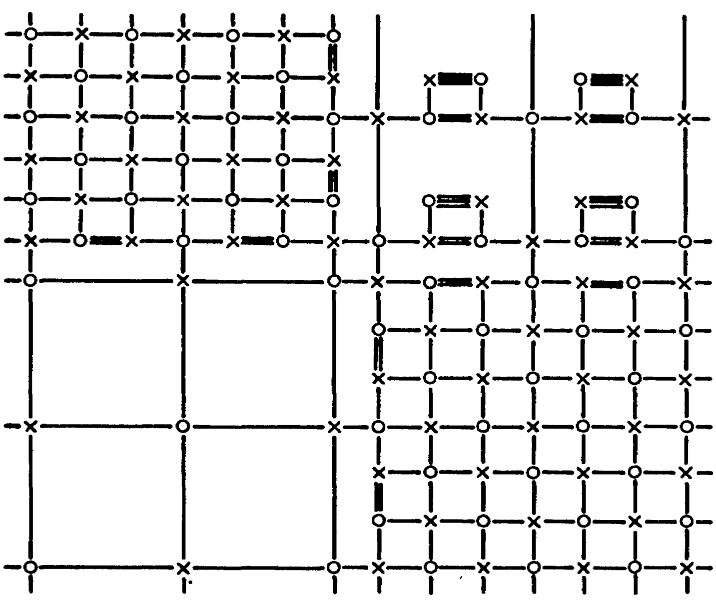

Figure 20

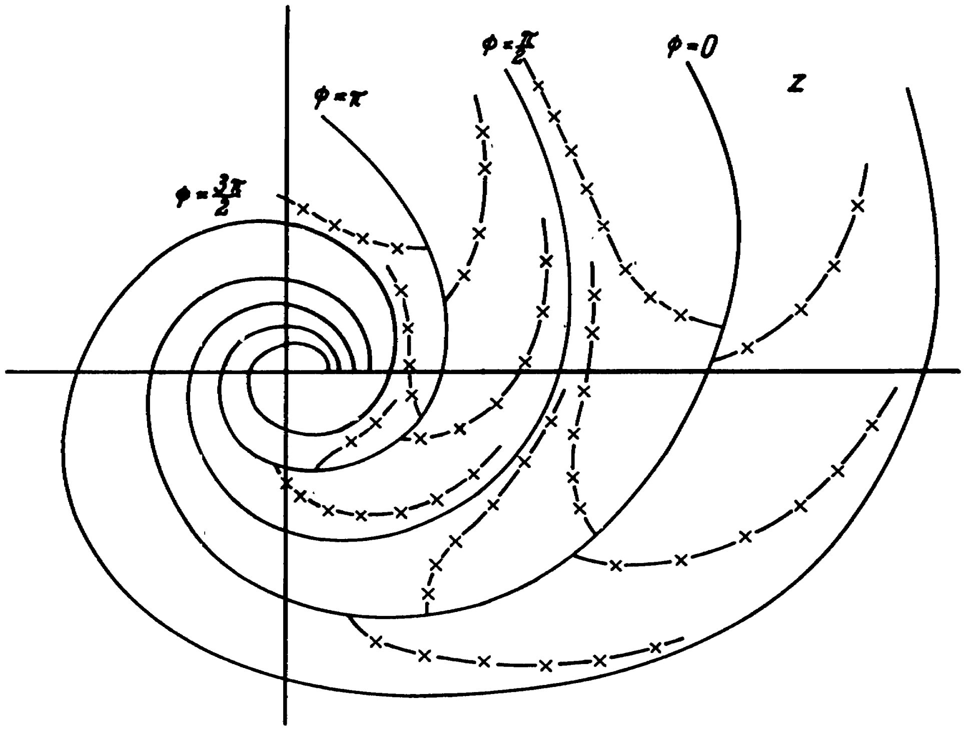

Figure 21

that the a-points lie on spirals. The uniformization over the Z-plane shows that in this plane the a-points lie over rays (Figure 19a). Figure 19b shows the distribution of a typical a-point in the z-plane.

Example 9. We conclude our examples by the graph of Figure 20. The figure consists of four quarter-ends with certain relative asymmetries. It is evident that the a-point distribution is given schematically by Figure 21.

REFERENCES

1. P. P. Belinsky, *Behavior of a quasi-conformal mapping at an isolated point*, Doklady Akad. Nauk SSSR(N.S.) 91 (1953), 709-710 (Russian).

2. G. Elfving, *Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung*, Acta Soc. Sci. Fennicae, N.S. 2, Nr. 3 (1934), 1-60.

3. H. P. Künzi, *Représentation et répartition des valeurs des surfaces de Riemann á extrémités bipériodiques*, C. R. Acad. Sci. Paris 234 (1952), 793-795.

4. ———, *Surfaces de Riemann avec un nombre fini d'extrémités simplement et doublement périodiques*, C. R. Acad. Sci. Paris 234 (1952), 1660-1662.

5. ———, *Über ein Teichmüllersches Wertverteilungsproblem*, Arch. Math. 4 (1953), 210-215.

6. ———, *Über periodische Enden mit mehrfach zusammenhängendem Existenzgebiet*, Math. Z. 61 (1954), 200-205.

7. ———, *Konstruktion Riemannscher Flächen mit vorgegebener Ordnung der Funktionen*, Math. Ann. 128 (1955), 471-474.

8. ———, *Zwei Beispiele zur Wertverteilungslehre*, Math. Z. 62 (1955), 94-98.

9. H. P. Künzi, *Neue Beiträge zur geometrischen Wertverteilungslehre*, Comment. Math. Helv. 29 (1955), 223-257.

10. ———, *Zur Theorie der Viertelsenden Riemannscher Flächen*, Comment. Math. Helv. 30 (1956), 107-115.

11. H. P. Künzi and H. Wittich, *Sur le module maximal de quelques fonctions transcendantes entières*, C. R. Acad. Sci. Paris 245 (1957), 1103-1106.

12. ———, *Sur la repártition des points ou certaines fonctions méromorphes prennent une valeur* a, C. R. Acad. Sci. Paris 245 (1957), 1991-1994.

13. R. Nevanlinna, *Eindeutige analytische Funktionen*, Second Ed., Springer, Berlin (1953).

14. ———, *Über fastkonforme Abbildungen* (lecture delivered at Helsinki in August 1957, not yet published).

15. A. Speiser, *Über Riemannsche Flächen*, Comment. Math. Helv. 2 (1930), 284-293.

16. O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. 3 (1938), 621-678.

17. ———, *Einfache Beispiele zur Wertverteilungslehre*, Deutsche Math. 7 (1944), 360-368.

18. E. Ullrich, *Flächenbau und Wachstumsordnung bei gebrochenen Funktionen*, Jber. Deutsch. Math Verein. 46 (1936), 232-274.

19. ———, *Zum Umkehrproblem der Wertverteilungslehre*, Nachr. Ges. Wiss. Göttingen I (N. F.) 1 (1940), 135-150 (1936).

20. H. Wittich, *Über die konforme Abbildung einer Klasse Riemannscher Flächen*, Math. Z. 45 (1939), 642-668.

21. ———, *Über die Wachstumsordnung einer ganzen transzendenten Funktion*, Math. Z. 51 (1947), 1-16.

22. ———, *Zum Beweis eines Satzes über quasikonforme Abbildungen*, Math. Z. 51 (1948), 278-288.

23. ———, *Über den Einfluss algebraischer Windungspunkte auf die Wachstumsordnung*, Math. Ann. 122 (1950), 37-46.

24. ———, *Bemerkung zur Wertverteilung von Exponentialsummen*, Arch. Math. 4 (1953), 202-209.

25. ———, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F. Heft 8 (1955), 1-163.

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