SOME OPAQUE SUBSETS OF A SQUARE

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We deal in this note with a fixed Euclidean plane. Let

$$Q = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

We say that a set S is *opaque*, if (i) S is a subset of Q, and (ii) every (straight) line that contains a point of Q also contains a point of S. If T is a subset of Q, the *distance set*, Δ_T, of T is defined to be the set of all real numbers d with the property that there exist points t, t' in T such that the distance between t and t' is d. Let J be the closed interval $[0, \sqrt{2}]$, and let $J^* = J - \{0, 1, \sqrt{2}\}$. If $T \subset Q$, then clearly $\Delta_T \subset J$, and if T is opaque, then $\Delta_T \supset \{0, 1, \sqrt{2}\}$ because T contains the vertices of Q.

A recent article by Sen Gupta and Basu Mazumdar [8] is devoted to showing that there exists a subset E of Q of first category and measure zero such that (a) every line that contains a point of Q and is not parallel to a side of Q also contains a point of E, and (b) $\Delta_E = J$, and if $0 < d < \sqrt{2}$ then there are infinitely many pairs of points in E such that the distance between the points of each pair is d. We remark that there is a very much simpler example of such a set E: the union of the two diagonals of Q not only satisfies (a) and (b), but is actually opaque, and is obviously a *nowhere dense* perfect subset of Q of measure zero.

There are perfect opaque sets that are even *punctiform* (that is, they contain no continuum having more than one point); in fact, Mazurkiewicz showed [6] that every polygon (to which we reckon interior points as well as frontier points) has a perfect punctiform subset that intersects every line that meets the polygon. We shall describe a perfect, punctiform, opaque set whose construction is akin to that of Mazurkiewicz but which is somewhat easier to see and to remember.

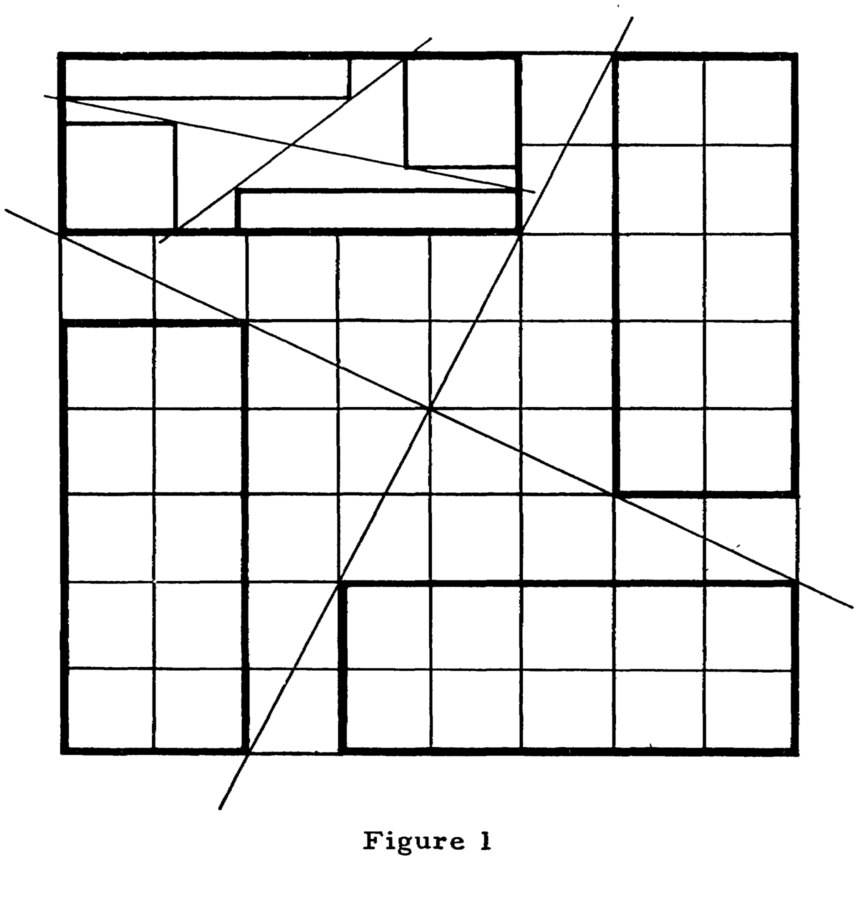

Figure 1

We begin (see Fig. 1) by dividing each side of Q into eight equal segments, thereby inducing a division

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of Q itself into 64 equal squares (think of a chessboard!). Now we single out four rectangles—those with the heaviest outlines in Fig. 1—each consisting of ten of these 64 equal squares. The union of these four rectangles we call R_1. It is obvious that if a line contains a point of Q then it contains a point of R_1. We divide each side of every one of the four rectangles constituting R_1 into eight equal segments, thereby inducing a division of each of these rectangles itself into 64 equal rectangles. As in the first stage of our construction, we combine 40 of these 64 rectangles into four rectangles; we thus obtain 16 rectangles in all, whose union we call R_2 (in Fig. 1, only four of these 16 new rectangles, those appearing in the upper left-hand rectangle of the first stage of our construction, are shown). It is obvious again that if a line contains a point of Q then it contains a point of R_2. Continuing in this manner, at the n^{th} stage of our construction we obtain 4^n rectangles, whose union we call R_n, such that if a line contains a point of Q then it contains a point of R_n. Clearly $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$. Let $F = \bigcap R_n$. Then F is perfect and punctiform [2, p. 93]. Let L be a line that intersects Q. Then, for every n, $R_n \cap L$ is compact and not empty, so that $F \cap L = \bigcap (R_n \cap L)$ is not empty [2, p. 56], and hence F is opaque.

Denjoy has indicated the existence of a perfect, punctiform, opaque set "of finite length" (perhaps the most successful treatment of his example is [3, p. 671]). Using Denjoy's main idea, we shall give another example in somewhat greater detail and show that the corresponding distance set is J.

Let $\{h_n\}$ be a sequence of positive numbers less than one and tending to zero as $n \to \infty$ (eventually h_n will be chosen suitably small). For every natural number n, we define, by induction, a set H_n, H_n being the union of 2^n closed (rectilinear) segments $S_{k_1k_2\ldots k_n}$, where k_j is either 0 or 1 ($j = 1, 2, \cdots, n$), obtained as follows (Fig. 2 shows the sets H_1, H_2, and H_3): Consider the diagonal of Q extending from $p_0 = (0, 1)$ to $q_1 = (1, 0)$. At the midpoint m of the diagonal, erect a vertical segment of length h_1, midpoint m, lower endpoint q_0, and upper endpoint p_1, and define S_{k_1} to be the segment extending from p_{k_1} to q_{k_1} ($k_1 = 0, 1$). Let n be a natural number, and suppose that H_n has been defined. At the midpoint $m_{k_1k_2\cdots k_n}$ of the segment $S_{k_1k_2\cdots k_n}$, erect a vertical segment of length h_{n+1}, midpoint $m_{k_1k_2\cdots k_n}$, lower endpoint $q_{k_1k_2\cdots k_n0}$, and upper endpoint $p_{k_1k_2\cdots k_n1}$, and define $S_{k_1k_2\cdots k_n0}$ to be the segment extending from the left endpoint of $S_{k_1k_2\cdots k_n}$ to $q_{k_1k_2\cdots k_n0}$, and $S_{k_1k_2\cdots k_n1}$ to be the segment extending from $p_{k_1k_2\cdots k_n1}$ to the right endpoint of $S_{k_1k_2\cdots k_n}$.

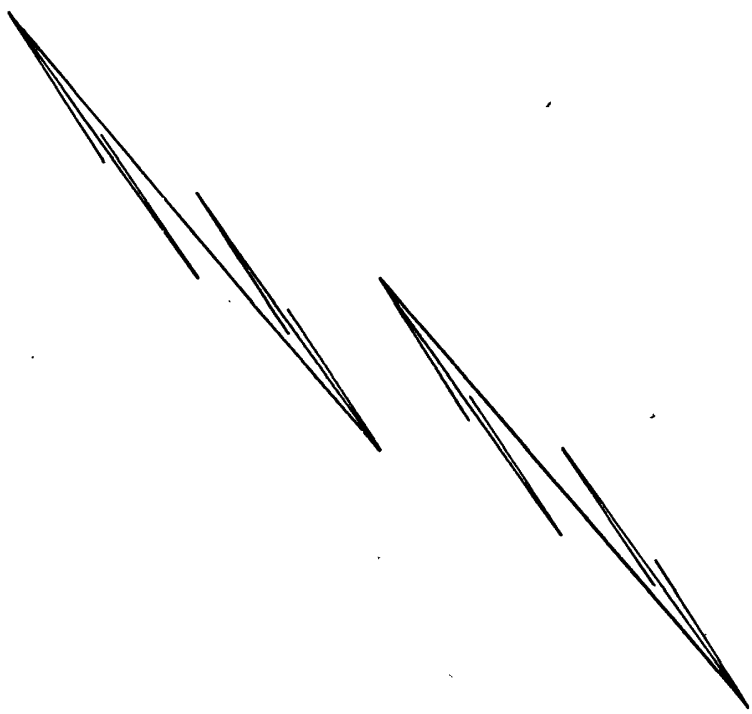

Figure 2

Now define D_1 to be the set $\overline{\lim}\, H_n$ [4, p. 104], D_2 to be the set obtained from D_1 by rotating the latter through 90^0 about the point (.5, .5), and, finally, D to be the set $D_1 \cup D_2$. If the numbers h_n ($n = 1, 2, 3, \cdots$) are chosen sufficiently small, D is a subset of Q.

According to [4, p. 105, 17·1·13], D_1 is closed. Not only does every one of the points $p_{k_1 k_2 \cdots k_n}$ and $q_{k_1 k_2 \cdots k_n}$ defined above belong to D_1, but, since the length of $S_{k_1 k_2 \cdots k_n}$ tends to zero as $n \to \infty$, it is readily seen that every element of D_1 is a limit point of such points of D_1. Thus D_1 is dense in itself, and hence D_1 is perfect; consequently, D is perfect.

If the definition of linear measure given by Carathéodory [1, p. 268] is applied to the set D_1, it is evident that the linear measure of D_1 can be made arbitrarily close to $\sqrt{2}$ by taking the numbers h_n ($n = 1, 2, 3, \cdots$) sufficiently small; and the linear measure of D can be made arbitrarily close to $2\sqrt{2}$.

It is clear from the definition of H_n, that D_1 is of dimension zero [4, p. 103]; hence [5, p. 18] D is of dimension zero, and is therefore [4, p. 103, 16·6·1] punctiform.

Let L be a line that contains a point of Q and is either vertical or has a nonnegative slope. Then, for every n, $H_n \cap L$ is obviously nonempty, so that [4, p. 107, 17·1·4] $\overline{\lim}\,(H_n \cap L)$ is nonempty; and since [4, p. 104, (1·2)]

$$D_1 \cap L = (\overline{\lim}\; H_n) \cap (\overline{\lim}\; L) \supset (H_n \cap L),$$

$D_1 \cap L$ is nonempty. After applying an analogous argument to D_2, we arrive at the conclusion that D is opaque.

We shall show that $\Delta_{D_1} = J$. The point (0, 1) belongs to D_1, and obviously $0 \in \Delta_{D_1}$. Let $0 < d \leq \sqrt{2}$, and consider a circle with center (0, 1) and radius d. This circle evidently intersects H_1, and if it intersects $S_{k_1 k_2 \cdots k_n}$ it also intersects $S_{k_1 k_2 \cdots k_n 0} \cup S_{k_1 k_2 \cdots k_n 1}$. Thus, for every natural number n, there exists a point $z_n \in H_n \subset Q$ such that the distance between z_n and (0, 1) is d. The sequence $\{z_n\}$ has at least one limit point z; $z \in D_1$, and the distance between (0, 1) and z is d. Hence $\Delta_{D_1} = J = \Delta_D$.

In view of the foregoing results, it might be conjectured that if S is opaque then $\Delta_S = J$; but this conjecture is false, as is shown by the following theorem.

THEOREM 1. *Let* $B \subset J^*$ *and* $|B| < 2^{\aleph_0}$. *Then there exists an opaque set* S *such that* $\Delta_S \subset J - B$.

Proof. We shall define the set S by transfinite induction. There are $2^{\aleph_0}$ lines that intersect Q in more than one point; well-order the set of these lines to form a transfinite sequence

$$L_0, L_1, L_2, \cdots, L_\xi, \cdots \qquad (\xi < \omega_\gamma),$$

where ω_γ is the initial number of $Z(2^{\aleph_0})$. Suppose that $\alpha < \omega_\gamma$ and that the point $p_\xi \in Q$ has been defined for every $\xi < \alpha$ in such a way that, if the set P_α consists of the vertices of Q and the distinct points in the sequence $\{p_\xi\}_{\xi<\alpha}$, then no number in B is the distance between any two points in P_α. If L_α contains a point $p \in P_\alpha$, define p_α to be one such point p. For the case where L_α contains no point in P_α, we note that, corresponding to each number b in B and each point p in P_α, the set $L_\alpha \cap Q$ contains at most two points at a distance b from p. Since $2 \cdot |\alpha| \cdot |B| < 2^{\aleph_0} = |L_\alpha \cap Q|$, there exists a point, say p_α, in $L_\alpha \cap Q$ such that no number in B is the distance between p_α and any point in P_α. The transfinite sequence $\{p_\xi\}_{\xi<\omega_\gamma}$ is now well defined. We denote by T the set of distinct points in this sequence, and by S the union of T with the set of vertices of Q. The set S clearly satisfies the conclusion of Theorem 1.

We shall show that under the continuum hypothesis there exist opaque sets having even thinner distance sets than the one described in Theorem 1.

THEOREM 2. *Assume that* $2^{\aleph_0} = \aleph_1$. *If* B *is a subset of* J* *of measure zero, then there exists an opaque set* S *such that* $\Delta_S \subset J - B$.

Proof. We define the set S by transfinite induction. Well-order the $\aleph_1$ lines that intersect Q in more than one point, to form a transfinite sequence

$$L_0, L_1, L_2, \cdots, L_\xi, \cdots \quad (\xi < \omega_1).$$

Suppose that $\alpha < \omega_1$ and that the point $p_\xi \in Q$ has been defined for every $\xi < \alpha$ in such a way that, if P_α consists of the vertices of Q and the distinct points in the sequence $\{p_\xi\}_{\xi<\alpha}$, then no number in B is the distance between any two points in P_α. If L_α contains a point $p \in P_\alpha$, define p_α to be one such point p. Suppose, however, that L_α contains no point in P_α. Let $p \in P_\alpha$, and denote by h the (positive) distance between p and L_α. Since B is of measure zero, it follows, by applying [7, p. 251, Theorem 3], that the set of real numbers

$$\{\sqrt{x^2 - h^2}: h \leq x < \sqrt{2},\ x \in B\}$$

is of measure zero. This implies that, if E_p is the set of points q in $L_\alpha \cap Q$ such that the distance between p and q is a number in B, then E_p is of measure zero. Since $|P_\alpha| \leq \aleph_0$, the set $\bigcup_{p \in P_\alpha} E_p$ is of measure zero. But $L_\alpha \cap Q$ is of positive measure, and therefore there exists a point, say p_α, in $L_\alpha \cap Q$ such that no number in B is the distance between p_α and any point in P_α. The transfinite sequence $\{p_\xi\}_{\xi<\omega_1}$ is now well defined. We denote by T the set of distinct points in this sequence, and by S the union of T with the set of vertices of Q. The set S clearly satisfies the conclusion of Theorem 2.

COROLLARY 1. *Assume that* $2^{\aleph_0} = \aleph_1$. *Then there exists a subset* C *of* J *of first category, and an opaque set* S, *such that* $\Delta_S = C$.

This follows from the fact that there exists a residual subset B of J* of measure zero.

THEOREM 3. *Assume that* $2^{\aleph_0} = \aleph_1$. *If* B *is a subset of* J* *of first category, then there exists an opaque set* S *such that* $\Delta_S \subset J - B$.

This can be proved by a category-theoretic argument so analogous to the measure-theoretic proof of Theorem 2 that we omit the details.

COROLLARY 2. *Assume that* $2^{\aleph_0} = \aleph_1$. *Then there exists a subset* C *of* J *of measure zero, and an opaque set* S, *such that* $\Delta_S = C$.

This follows from the fact that there exists a subset B of J* of first category such that J* - B is of measure zero.

Remark 1. An analysis of the proof of Theorem 2 shows that the assumption that $2^{\aleph_0} = \aleph_1$ can be replaced in that theorem and in Corollary 1 by the assumption that the linear continuum is not the union of fewer than $2^{\aleph_0}$ linear sets of measure zero. Similarly, the assumption that $2^{\aleph_0} = \aleph_1$ can be replaced in Theorem 3 and Corollary 2 by the assumption that the linear continuum is not the union of fewer than $2^{\aleph_0}$ linear sets of first category.

Remark 2. If S is an opaque set, is it necessary that Δ_S be everywhere dense in J? If so, then $\Delta_T = J$ for every closed opaque set T.

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Added in proof. Remark 3. If S is a linearly measurable opaque set, how small can the linear measure of S be? Since the orthogonal projection of S onto a diagonal of Q is that diagonal, the linear measure of S is at least $\sqrt{2}$ (see W. Gross, *Über das Flächenmass von Punktmengen,* Monatsh. Math. Phys. 29 (1918), 145-176); and since the orthogonal projection of S onto any line has linear measure at least one, the linear measure of S is greater than $\pi/2$ (see H. G. Eggleston, *Problems in Euclidean space: application of convexity,* New York, 1957, p. 35).

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