AN INTEGRAL INVOLVING THE ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

F. M. Ragab

1. INTRODUCTION

The formula

$$(1)\qquad \begin{aligned}&\int_0^\infty (\sinh u)^{m+1} P_n^{-m}(\cosh u)(\cosh u)^{-\ell} E(p;\alpha_r:q;\rho_s: z\cosh^2 u)\,du\\ &= 2^{-m-1} E\left[\begin{matrix}\alpha_1,\cdots,\alpha_\rho,\frac{\ell-m+n}{2},\frac{\ell-m-n-1}{2};\\ \rho_1,\cdots,\rho_q,\frac{\ell}{2},\frac{\ell+1}{2};\end{matrix}\; z\right]\end{aligned}$$

will be proved under the conditions

$$\Re(\ell - m + n) > 0, \qquad \Re(\ell - m - n) > 1, \qquad \Re(m) > -1;$$

the functions P_n^{-m} and E under the integral sign are defined by the relations

$$(2)\qquad P_n^{-m}(t) = \frac{2^{-m}(t^2-1)^{m/2}t^{n-m}}{\Gamma(m+1)}\,{}_2F_1\left[\begin{matrix}\frac{m-n+1}{2},\frac{m-n}{2};\\ m+1;\end{matrix}\;\frac{t^2-1}{t^2}\right],$$

$$(3)\qquad E(p;\alpha_r:q;\rho_s:z) = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\rho_1)\cdots\Gamma(\rho_q)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \rho_1,\cdots,\rho_q;\end{matrix}\;\frac{-1}{z}\right]\qquad (p\le q);$$

for a discussion of the E-functions, see [3, p. 352].

From formula (1), integration formulas involving many special functions (for example, those of Bessel, Whittaker, Struve) can be deduced. The formulas thus obtained provide information on the asymptotic behavior of the integrals for large values of $|z|$; for the asymptotic expansion of the E-functions is given by MacRobert [3, p. 358], and the other functions that occur can be expressed in terms of ordinary generalized hypergeometric functions whose asymptotic expansion has been investigated by several writers (Barnes [1], Wright [5], Meijer [4]).

Received February 21, 1957.

2. PROOF OF THE FORMULA

We shall make use of the known relation

$$\int_0^1 x^{\gamma-1} (1-x)^{\rho-1} {}_2F_1(\alpha, \beta; \gamma; x)\, dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}, \tag{4}$$

where $\Re(\gamma) > 0$, $\Re(\rho) > 0$, $\Re(\gamma + \rho - \alpha - \beta) > 0$ (see [2, p. 399, Formula (4)]). Let I denote the integral which is the left member of equation (1). In I, we put $\cosh u = (1-x)^{-1/2}$ and use (2) and (3) to obtain

$$I = \frac{2^{-m-1}\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(m+1)\Gamma(\rho_1)\cdots\Gamma(\rho_q)}\int_0^1 x^m (1-x)^{(\ell-m-n-3)/2}$$

$$\times {}_2F_1\left[\begin{matrix}\frac{m-n-1}{2}, \frac{m-n}{2}; x\\ m+1;\end{matrix}\right] \cdot {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p;\\ \rho_1, \cdots, \rho_q;\end{matrix} -(1-x)/2\right] dx.$$

We next insert for ${}_pF_q$ its power series expansion, and we reverse the order of integration and summation to write I in the form

$$I = \frac{2^{-m-1}\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(m+1)\Gamma(\rho_1)\cdots\Gamma(\rho_q)}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha_1)_k\cdots(\alpha_p)_k}{z^k(\rho_1)_k\cdots(\rho_q)_k}\cdot B, \tag{5}$$

where

$$B = \int_0^1 x^m (1-x)^{(2k+\ell-m-n-3)/2}\, {}_2F_1\left[\begin{matrix}\frac{m-n+1}{2}, \frac{m-n}{2}; x\\ m+1\end{matrix}\right] dx.$$

The integral B is calculated by formula (4):

$$B = \frac{\Gamma\left(\frac{2k+\ell-m-n-1}{2}\right)\Gamma\left(\frac{2k+\ell-m+n}{2}\right)}{\Gamma\left(\frac{2k+\ell}{2}\right)\Gamma\left(\frac{2k+\ell+1}{2}\right)}$$

$$= \frac{\left(\frac{\ell-m-n-1}{2}\right)_k\left(\frac{\ell-m+n}{2}\right)_k}{\left(\frac{\ell}{2}\right)_k\left(\frac{\ell+1}{2}\right)_k}\cdot\frac{\Gamma\left(\frac{\ell-m-n-1}{2}\right)\Gamma\left(\frac{\ell-m+n}{2}\right)}{\Gamma\left(\frac{\ell}{2}\right)\Gamma\left(\frac{\ell+1}{2}\right)}.$$

The insertion of the value of B into the right member of equation (5) at once yields the desired result (1).

REFERENCES

1. E. W. Barnes, *The asymptotic expansion of integral functions defined by generalised hypergeometric series*, Proc. London Math. Soc. (2) 5 (1907), 59-116.

2. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of integral transforms, Vol. II. Based, in part, on notes left by Harry Bateman,* McGraw-Hill, New York, 1954.

3. T. M. MacRobert, *Functions of a complex variable*, Fourth Edition, Glasgow, 1954.

4. C. S. Meijer, *On the G-function, VIII,* Nederl. Akad. Wetensch. Indag. Math. 8 (1946), 713-723.

5. E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. 10 (1935), 286-293.

Institute for Advanced Study, U.S.A.
and
University Abassia, Cairo, Egypt