THE FOURIER SERIES OF FUNCTIONS OF BOUNDED pTH POWER VARIATION

Fu Cheng Hsiang

1. A function $f(x)$ is said to be of bounded pth power variation in $[a, b]$, or to be of Wiener class $W_p(a, b)$, provided the upper bound $V_p(f; a, b)$ of the expression

$$(\sum_r |f(x_r) - f(x_{r-1})|^p)^{1/p},$$

taken with respect to all subdivisions $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, is finite. We state some relevant known facts as lemmas.

LEMMA 1 (see [3, p. 259]). *The quantity* $V_p(f; a, b)$ *is a decreasing function of* p, *and for all* $q \geq p$,

$$V_p(f; a, b) \leq V_p^{q/p}(f; a, b)[\operatorname{Osc}(f; a, b)]^{(q-p)/q},$$

where $\operatorname{Osc}(f; a, b)$ *denotes the oscillation of* $f(x)$ *in* (a, b).

In connection with Stieltjes integration and functions of class W_p, L. C. Young has established the following inequality of Hölder type [3, p. 266].

LEMMA 2. *Let* $f \in W_p(a, b)$ *and* $g \in W_q(a, b)$, *where* $p > 1$, $q > 1$, *and* $s = 1/p + 1/q > 1$. *If* f *and* g *have no common discontinuity in* (a, b), *then the Riemann-Stieltjes integral* $\int_a^b f(x)\,dg(x)$ *exists, and for each* η *in* $[a, b]$,

$$\left|\int_a^b [f(x) - f(\eta)]\,dg(x)\right| \leq [1 + \zeta(s)]\, V_p(f; a, b)\, V_p(g; a, b),$$

where $\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}$.

LEMMA 3. *If* $p > 1$, $f \in W_p(a, b)$, $g \in W_p(a, b)$, *and* $h(x) = f(x)g(x)$, *then* $h(x) \in W_p(a, b)$.

Proof. By hypothesis, $|f(x)|$ and $|g(x)|$ are bounded by some constant K. On writing

$$f(x_r)g(x_r) - f(x_{r-1})g(x_{r-1}) = [f(x_r) - f(x_{r-1})]g(x_r) + f(x_{r-1})[g(x_r) - g(x_{r-1})]$$

and applying Minkowski's inequality, we obtain the relation

$$(\sum_r |f(x_r)g(x_r) - f(x_{r-1})g(x_{r-1})|^p)^{1/p}$$

$$\leq K(\sum_r |f(x_r) - f(x_{r-1})|^p)^{1/p} + K(\sum_r |g(x_r) - g(x_{r-1})|^p)^{1/p}.$$

The lemma now follows on taking the upper bound on both sides.

Received July 10, 1957.

2. We suppose further that f is an L-integrable function, periodic with period 2π. Let

$$f(t) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum A_n(t),$$

and write

$$\phi(t) = \frac{1}{2}[f(x+t) + f(x-t)] - s, \qquad \Phi(t) = \int_0^t \phi(u)\,du.$$

For $0 < \alpha < 1$, the question whether $\Sigma A_n(t)$ is Cesàro summable $(C, -\alpha)$ at the point $t = x$ does not depend solely on the local properties of f near x. But Bosanquet and Offord [1, p. 274] have established the following proposition.

LEMMA 4. *Necessary and sufficient conditions for the series* $\Sigma A_n(x)$ *to be summable* $(C, -\alpha)$ *to* s *are that*

(i) $$A_n = o(n^{-\alpha}),$$

(ii) $$\Phi(t) = o(t) \qquad (t \to +0),$$

(iii) $$\lim_{\delta \to 0} \limsup_{n \to \infty} \left| n^{\alpha} \int_{1/n}^{\delta} \phi(t)(1 - t/\delta) \frac{\sin(n, t)}{t^{1-\alpha}}\,dt \right| = 0,$$

where δ *is an arbitrary positive constant, and* $\sin(n, t) = \sin((n + 1/2 - \alpha/2)t + \alpha\pi/2)$.

3. The purpose of the present note is to improve the following theorem of Bosanquet and Offord [1, p. 280].

THEOREM A. *Let* $\eta > 0$, $0 < \alpha < 1$. *If* (i) *and*

(iv) $$\int_0^t |d\{u\phi(u)\}| = O(t) \qquad (0 < t < \eta)$$

are satisfied, and if $\Sigma A_n(t)$ *is summable* (A) *to* s *at* $t = x$, *then it is summable* $(C, -\alpha)$ *to* s *at* $t = x$.

Our result is as follows.

THEOREM. *Let* $p > 0$, $0 < \alpha < 1$, $\alpha p < 1$. *If*

(1) $$A_n = o(n^{-\alpha}),$$

(2) $$V_p(\phi;\, t, 2t) = O(1)$$

as $t \to +0$, *and the limit* $s = \frac{1}{2}[f(x+0) + f(x-0)]$ *exists, then* $\Sigma A_n(x)$ *is summable* $(C, -\alpha)$ *to* s.

(For $p = 1$, condition (2) is equivalent to (iv); see [2]).

Proof. Since the limit $f(x+0) + f(x-0)$ exists, condition (ii) of Lemma 4 holds. In order to prove the theorem, we need only to establish condition (iii) of the same lemma. We write

$$J = n^{\alpha}\int_{1/n}^{\delta} \phi(t)\left(1 - \frac{t}{\delta}\right)\frac{\sin(n, t)}{t^{1-\alpha}}\,dt = \int_{1/n}^{\delta} \phi(t)\left(1 - \frac{t}{\delta}\right)dg_n^{\alpha}(t),$$

where

$$g_n^{\alpha}(t) = n^{\alpha}\int_0^t \frac{\sin(n, u)}{u^{1-\alpha}}\,du.$$

We shall now show that if $q > 1/(1 - \alpha)$, then there exists a positive number $C = C(q, \alpha)$ such that $V_q(g_n^{\alpha}; 0, \pi) \leq C$. Let

$$a_0 = 0, \qquad a_m = \frac{(m - \alpha/2)\pi}{n - \alpha/2 + 1/2} \quad (m = 1, 2, \cdots, n), \qquad a_{n+1} = \pi.$$

Then $0 = a_0 < a_1 < \cdots < a_{n+1} = \pi$, and $\sin(n, t)$ has constant sign, in each of the intervals (a_{m-1}, a_m); therefore, if $a_{m-1} \leq x_1 \leq x_2 \leq \cdots \leq x_r \leq a_m$, then

$$\sum_k |g_n^{\alpha}(x_k) - g_n^{\alpha}(x_{k-1})|^q \leq \left(\sum |g_n^{\alpha}(x_k) - g_n^{\alpha}(x_{k-1})|\right)^q = n^{\alpha}\left|\int_{a_{m-1}}^{a_m} \frac{\sin(n, t)}{t^{1-\alpha}}\,dt\right|^q.$$

It follows that

$$V_q^q(g_n^{\alpha}; 0, \pi) = O\left(\sum (m + 1)^{q(\alpha-1)}\right).$$

In particular, for any subinterval $[a, b]$ of $[0, \pi]$,

$$V_q^q(g^{\alpha}; a, b) \leq K \sum_{m=m_1}^{m_2} m^{q(\alpha-1)},$$

where $m_1 = [na/\pi] - 1$, $m_2 = [nb/\pi] - 1$ and K is a numerical constant.

To each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\mathrm{Osc}(\phi; 0, \delta) < \varepsilon$ and $\max(|\phi|; 0, \delta) < \varepsilon$. Fixing δ, we increase p slightly so that $p > 1$, but so that the inequality $\alpha p < 1$ still holds. On applying Lemmas 1 and 3, we see that

$$V_p(\phi(t)(1 - t/\delta); t, 2t) < \varepsilon_1$$

for $0 \leq 2t \leq \delta$, where $\varepsilon_1 \to 0$ as $\varepsilon \to 0$. Having chosen p, we choose q so that

$$q(1 - \alpha) > 1, \qquad s = 1/p + 1/q > 1.$$

If $2a = 2^{\nu+1}/n \leq \delta$, then, by Lemma 2,

$$I_\nu = \left|\int_a^{2a} \phi(t)\left(1 - \frac{t}{\delta}\right)dg_n^{\alpha}(t)\right|$$

$$\leq \left|\int_a^{2a} \phi(t)\left(1 - \frac{t}{\delta}\right) - \phi(\eta_\nu)\left(1 - \frac{\eta_\nu}{\delta}\right)dg_n^{\alpha}(t)\right| + \left|\phi(\eta_\nu)\left(1 - \frac{\eta_\nu}{\delta}\right)\int_a^{2a} dg_n^{\alpha}(t)\right|$$

$$(a \leq \eta_\nu \leq 2a)$$

$$\leq (1 + \zeta(s)) V_p \left(\phi(t) \left(1 - \frac{t}{\delta}\right); a, 2a\right) V_q(g_n^\alpha; a, 2a) + \varepsilon O(2^{\nu(\alpha-1)})$$

$$\leq \varepsilon_1 (1 + \zeta(s)) V_q(g_n^\alpha; a, 2a) + \varepsilon O(2^{\nu(\alpha-1)})$$

$$\leq \varepsilon_1 (1 + \zeta(s)) \left(\sum_{m > 2^\nu} m^{q(\alpha-1)}\right)^{1/q} + \varepsilon O(2^{\nu(\alpha-1)}).$$

Let N be the greatest integer such that $2^{N+1}/n \leq \delta$. Then

$$\left|\int_{1/n}^{\delta} \phi(t)\left(1 - \frac{t}{\delta}\right) dg_n^\alpha(t)\right| \leq \sum_{\nu=0}^{N+1} I_\nu + \left|\int_{\frac{2^{N+1}}{n}}^{\delta} \phi(t)\left(1 - \frac{t}{\delta}\right) dg_n^\alpha(t)\right|$$

$$= \varepsilon_1 O\left(\sum_{\nu=0}^{N+1} 2^{\nu Q}\right) + \varepsilon O\left(\sum_{\nu=0}^{N} 2^{\nu(\alpha-1)}\right),$$

where $Q = 1/q - (1 - \alpha) < 0$. It follows immediately that $\lim_{\delta \to 0} \lim_{n \to \infty} \sup J = 0$. This proves the theorem.

REFERENCES

1. L. S. Bosanquet and A. C. Offord, *A local property of Fourier series*, Proc. London Math. Soc. (2) 40 (1936), 273-280.

2. G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* (VII): *On Young's convergence criterion for Fourier series*, Proc. London Math. Soc. (2) 28 (1928), 301-311.

3. L. C. Young, *An inequality of Hölder's type connected with Stieltjes integration*, Acta Math. 67 (1936), 251-282.

National Taiwan University
Taipeh, China