A THEOREM ON NONLOXODROMIC MÖBIUS TRANSFORMATIONS

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If a Möbius transformation has one fixed circle, it has a coaxal family of fixed circles. The theorem I shall prove relates the mapping on any fixed circle to the axis of this family, using the real projective structure of the plane.

THEOREM 1: *If* g *is a fixed circle of a Möbius transformation* T *and the distinct points* A, B, C *have images* A', B', C' *on* g, *then the axis of the system of fixed circles of* T *is the Pascal line of the hexagon* AB'CA'BC'.

The proof makes use of the theory of projectivities on a conic (see [1], Chap. VII). This theory is based upon the fact that if A, B, C, D are points on a conic and P is a variable point on it, then the cross-ratio (PA PB, PC PD) is constant. Using this four-point invariant, we can define projective correspondences on the conic, and it is clear that any projectivity on a line in the plane of the conic can be mapped onto the conic from any point of the latter with preservation of its character: that is, fixed points map onto fixed points, and the invariant cross-ratio of a point and its image with respect to the fixed points is preserved. A pretty result of this theory is the following.

THEOREM 2. *If* J *is a projectivity on a conic and* A', B', C' *are the images of distinct points* A, B, C *under* J, *then the Pascal line of the hexagon* AB'CA'BC' *meets the conic in the fixed points of* J.

For a real conic this theorem classifies the projectivity: J is hyperbolic, parabolic or elliptic according as the line cuts the conic in 2, 1 or 0 points. The classification of nonloxodromic Möbius transformations by the relation of the fixed axis to the fixed circles is analogous (see [2], Chap. 1).

Let us consider Theorem 1 in the hyperbolic case, the distinct fixed points being M and N. The family G of fixed circles is the family on M and N; the axis u is the line MN. The cross-ratio (Z T(Z), M N) = λ is real. To see that the transformation on g is a projectivity in the sense of the above discussion, we note that the mapping on u is a real projectivity, and that if P is a point of g other than M or N, then (PZ PT(Z), PM PN) = λ for Z either on u or on g. Thus the pencil at P relates the transformations as required; the circle is a conic on which Theorem 2 holds, and Theorem 1 follows for this case.

If T is parabolic, u is the tangent to g at the single fixed point M. We may argue as before if, instead of λ as defined above, we use the fact that in this case (Z T^2(Z), T(Z) M) = -1, this being necessary and sufficient for a projectivity (real or complex) to have no other fixed point.

In the elliptic case we proceed differently. Here $\lambda = e^{i\theta}$, and u is the perpendicular bisector of the segment MN. Let W be the coaxal system on M and N; all circles of W are then orthogonal to g, and M and N are the "limiting points" of G. Let us say M lies within g; the angle at M between the arc MA and its image arc MA' is θ.

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To relate the circle geometry to the rectilinear structure of the plane, we observe from elementary considerations that the common chords of g and individual circles of W are concurrent at Q, the pole of u. Let the center of g be E. Now let V be a circular transformation which preserves g and takes M to E, the arcs through M thus mapping onto diameters. To such a V there corresponds a unique real collineation H which preserves g and takes Q to E, while the two mappings on g are identical. This is evident if we regard g as a conformal (Poincaré) model of the hyperbolic plane, and simultaneously as a projective (Cayley-Klein) model; but only rudimentary considerations are required. Since H treats the plane as projective, the image of u under H, being the polar of E, must be regarded as a *line* at infinity. Let the images of A, B, C under V (or H) be $\bar{A}$, $\bar{B}$, $\bar{C}$. Since V preserves angles, the angles $\bar{A}E\bar{A}'$, $\bar{B}E\bar{B}'$, $\bar{C}E\bar{C}'$ are each θ. Now the transformed Pascal line is determined by the intersections $\bar{A}\bar{B}'$, $\bar{A}'\bar{B}$, and so forth. But the equality of the stated angles makes these lines parallel; hence the Pascal line of the transformed hexagon is the transformed u, which concludes the proof.

REFERENCES

1. H. S. M. Coxeter, *The real projective plane*, McGraw-Hill, New York, 1949.

2. L. R. Ford, *Automorphic functions*, McGraw-Hill, New York, 1929.

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