A NOTE ON THE SYSTEM GENERATED BY A SET OF ENDOMORPHISMS OF A GROUP

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The study of a set $\mathfrak{E}$ of endomorphisms of a group G has been limited generally to the case of an abelian group G, although the set $\mathfrak{E}_1$ of all normal endomorphisms of a nonabelian group G has been studied by Fitting and others (see [3], [4]). In the abelian case, a ring R can be formed from $\mathfrak{E}$ and studied instead of $\mathfrak{E}$. In similar fashion a type of near-ring (a distributively-generated near-ring) N can be formed from $\mathfrak{E}$ in the general case, and it is the purpose of this note to develop a structure theory for these near-rings which generalizes the Artin-Wedderburn theory for rings. The development is kept brief, both because of the analogy and because of the existence of some information on general near-rings (see [1], [2], [5]). Certain distinctions between the rings and the non-ring near-rings are discussed in the final section.

1. PRELIMINARY REMARKS

Let $\mathfrak{E}$ be a set of endomorphisms of an additively-written group G which satisfies the DCC on $\mathfrak{E}$-subgroups. Addition and multiplication of endomorphisms E and F of G are defined by the equations

$$g(E + F) = gE + gF \quad \text{and} \quad g(EF) = (gE)F \quad (g \in G).$$

Extend the set $\mathfrak{E}$ to the semigroup $\mathfrak{E}'$ of all products of finitely many elements of $\mathfrak{E}$. Then the subset $R(\mathfrak{E})$ of the set of all mappings of G into itself consisting of all finite linear combinations $\Sigma r_i E_i$ of elements E_i of $\mathfrak{E}'$ with rational integral coefficients r_i will be called the *system generated by* $\mathfrak{E}$.

Now a *near-ring* N is a set of elements with two binary operations, written as addition and multiplication, such that

i) N is a group relative to addition;

ii) N is a semigroup relative to multiplication;

iii) $a(b + c) = ab + ac$ for all $a, b, c \in N$.

An additive subgroup M of N is called a *right module* provided $MN \subseteq M$. A near-ring N which

i) contains a multiplicative semigroup D of right distributive elements d ($(b + c)d = bd + cd$ for all $b, c \in N$) such that each element of N can be written as a finite linear combination $\Sigma r_i d_i$ of d_i of D with rational integral coefficients r_i,

ii) satisfies the DCC for right modules

is called a *distributively-generated* near-ring (DGN-ring). Obviously $R(\mathfrak{E})$ is a DGN-ring. If the additive group of a DGN-ring is denoted by G and D by $\mathfrak{E}$, then clearly the system $R(\mathfrak{E})$ is a homomorphic image of N (right regular representation) and is isomorphic with N if, for instance, N has a multiplicative identity.

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Henceforth only DGN-rings will be considered, and the following information about N will be needed.

1) An *homomorphism* of N is an operation-preserving mapping π of N. Evidently $\pi(N)$ is a DGN-ring. For the usual reason, an *ideal* T of N is defined to be a normal additive subgroup of N such that $TN \subseteq T$, $NT \subseteq T$. This leads to a biunique correspondence between the ideals of N and its homomorphisms.

2) $0n = n0 = 0$, where 0 is the additive identity of N and n is an arbitrary element of N.

3) N is a ring if and only if N is an abelian group relative to addition.

4) If N contains two ideals A and B such that $A \cap B$ contains only 0 and if each $n \in N$ is expressible as $n = a + b$ ($a \in A$, $b \in B$), then N is the *direct sum* $A \oplus B$ of A and B.

2. THE RADICAL

An ideal T of N is said to be *regular* if the difference DGN-ring N - T has a multiplicative identity, and *left-regular* if N - T has a left multiplicative identity.

The *radical* R of a DGN-ring is defined to be the intersection of all maximal left-regular ideals of N, or to be N itself if no such ideals exist. (Regularity could be used also, as we shall see.) A *semisimple* DGN-ring is a DGN-ring whose radical is the zero ideal. A *simple* DGN-ring is a semisimple DGN-ring without proper ideals.

THEOREM 1. $\overline{N} = N - R$ *is semisimple and expressible uniquely as a direct sum of simple* DGN-*rings* N_i,

$$\overline{N} = N_1 \oplus \cdots \oplus N_r .$$

Each N_i *has the left identity* e_i, *and* $e = e_1 + \cdots + e_r$ *is the left identity of* $\overline{N}$.

The proof is straight-forward, and we omit it.

THEOREM 2. *The left identity* e *of a semisimple* DGN-*ring* N *is also a right identity element.*

It is only necessary to prove the theorem for the case where N is simple. We form the set S of the elements $a - ae$ for all $a \in N$. Since N is distributively-generated, each element of S annihilates N from the left. Therefore S either consists of the element 0 or it generates a two-sided ideal, which contradicts the fact that N has a left identity. Thus $ae = a = ea$ for each a in N.

A consequence of this result is that a simple DGN-ring is also a simple near-ring in the sense of Blackett [1]. Therefore we can utilize certain of his results on simple near-rings.

We note that the approach outlined here can be modified slightly and applied to near-rings possessing the DCC on right modules. The results are essentially the same.

3. SIMPLE DGN-RINGS

Throughout this section, N will denote a simple DGN-ring with unit element e. From the work of Blackett it follows that e decomposes into mutually orthogonal idempotents $e_1, \cdots, e_n$ such that $e_i N$ is a normal additive subgroup of N. Evidently $e_i N \cdot N \subseteq e_i N$. (A normal additive subgroup with this property is called a *right ideal* of N.) Moreover, $e_i N$ is a minimal right module, and since N is the sum of these right ideals, each element a of N is expressible uniquely, except for ordering, as a sum

$$a = a_1 + \cdots + a_n \qquad (a_i \in e_i N).$$

As in the theory of rings, we now consider the n^2 sets $e_i N e_j$.

THEOREM 3. *The nonzero elements of* $e_i N e_i$ *form a multiplicative group* N_i. N *contains* n^2 *"matric units"* c_{ij}, $c_{ii} = e_i$, $c_{ij} c_{k\ell} = \delta_{jk} c_{i\ell}$, *which provide a one-to-one correspondence between the* $e_i N e_j$. *Furthermore, the group* N_i *is isomorphic with the group of all automorphisms of* $e_i N$ *(as an additive group) which commute with the elements of* N *interpreted as right operators on* $e_i N$.

(In an unpublished article, Prof. H. Wielandt has proved a similar theorem for near-rings which are not rings and which possess a primitive right representation group.)

(i) If $a \in N_i$, then $a = e_i a e_i$. Now $(e_i a e_i) e_i N$ is a right module of N contained in $e_i N$. Hence $e_i a e_i N = e_i N$, and there exists an x in $e_i N$ such that $e_i a e_i x = e_i$. Therefore $e_i x e_i \in N_i$ is the inverse of a.

(ii) As before, $(e_i N e_j) e_j N = e_i N$. Hence there exist elements c_{ij} in $e_i N e_j$ and c_{ji} in $e_j N e_i$ such that $c_{ij} c_{ji} = e_i$. Now $c_{ji}(e_i N e_i) c_{ij} \subseteq e_j N e_j$. Hence

$$c_{ij}(e_j N e_j) c_{ji} = e_i N e_i .$$

Similarly, there exist elements d_{ij} in $e_i N e_j$ and d_{ji} in $e_j N e_i$ such that $d_{ji} d_{ij} = e_j$ and $d_{ji}(e_i N e_i) d_{ij} = e_j N e_j$. Since $d_{ji} c_{ij} = a \neq 0$ in N_j and $d_{ji} c_{ij} c_{ji} = d_{ji} = a c_{ji}$ while $d_{ij} = c_{ij} a^{-1}$, it follows that the matric units may be selected as stated.

(iii) If $e_i a e_i \in N_i$, denote by α the mapping $x \to (e_i a e_i) x = \alpha(x)$ $(x \in e_i N)$. Clearly, α is an automorphism of $e_i N$, and $\alpha n = n \alpha$ for each element n of N. Conversely, if α is such an automorphism of $e_i N$, then its effect on $e_i N$ is determined by its effect on e_i. Since it sends e_i onto $e_i a e_i$, it is evident that α corresponds to $e_i a e_i$ (see [5], pp. 76, 77).

Now we shall consider the differences between the simple DGN-rings which are rings and those which are not.

THEOREM 4. N *is a ring if and only if each* $e_i N e_j$ *is a commutative group relative to addition.*

This result is immediate.

THEOREM 5. *If each element of* $e_i N$ *is uniquely expressible as*

$$e_i r_1 + \cdots + e_i r_n \quad (r_j \in e_i N e_j),$$

then $e_i N e_i$ *is a* DN-*ring* F, *and* N *is* F_n, *the set of all* n-*by*-n *matrices with elements in* F.

We must consider the additive group $e_i N$ and show that each $e_i N e_j$ is an additive group. Now if $e_i r = \Sigma_j e_i r_j$, then the $e_i r_j$ are pairwise commutative, for the idempotents $e_1, \cdots, e_n$ commute with each other since the $e_i N$ are right ideals. Therefore

$$e_i r = e_i re = e_i r \left(\sum e_j\right) = e_i r \left(\sum e_{\pi(j)}\right).$$

Let $C_{ij} = C(e_i Ne_j)$ be the set of all elements of $e_i N$ which commute with each element of $e_i Ne_j$ relative to addition. Then C_{ij} is an additive group, and it follows simply that if $C_{ij} \cap e_i Ne_j$ contains more than one element, then $e_i Ne_j$ is an abelian group. For this intersection is in the center of $e_i N$, and $e_i N$ has a nontrivial center only if it is an abelian group. If $C_{ij} \cap e_i Ne_j$ contains only 0 for some j, then it contains only 0 for each j. Clearly $e_i Ne_j = \bigcap_{k \neq j} C(e_i Ne_k)$, and hence it is an additive group. In either case, each $e_i Ne_i$ is a DN-ring F (or a *near-field;* [5, p. 76]), and it is evident that N can be written as the set of all n-by-n matrices over F (although in the second case these matrices will not have all the properties of matrices over ordinary division rings).

The concept of characteristic may be introduced for DN-rings. A DN-ring has *characteristic* $m \neq 0$ if $me = e + \cdots + e = 0$ (e the identity) and m is the least positive rational integer with this property. If no such integer exists, the characteristic is defined to be 0. Evidently, $m \neq 0$ must be a prime p.

THEOREM 6. *If each element of* $e_i N$ *is uniquely expressible as* $e_i r_1 + \cdots + e_i r_n$ $(e_i r_j \in e_i Ne_j)$ *and if* $e_i N$ *contains finitely many elements, then* N *is a central simple ring.*

The DN-ring $e_i Ne_i$ must have characteristic p, which means that $e_i N$ is an additively-written p-group. Since a p-group possesses a nontrivial center, N is a ring, and the result follows.

In general, if N is finite, the following three statements are true.

(i) Each nonzero element of $e_i Ne_j$ has the same additive order d_{ij}.

(ii) If $d_i = \text{l. c. m.}_j\, d_{ij}$, then $d_1 = \cdots = d_n = d$ is the order of N in that $dr = r + \cdots + r = 0$ for each r in N and d is the least positive rational integer with this property.

(iii) $d = \text{l. c. m.}\, d_{ii}$.

THEOREM 7. *If each* $e_i Ne_i$ *is a finite additive group, then* N *is a central simple ring.*

As we have seen, $e_i Ne_i$ is a DN-ring of characteristic p_i. Hence $e_i Ne_i$ contains $p_i^{x_i}$ elements. But each $e_i Ne_i$ contains the same number of elements, so that $p_i = p$ and $e_i N$ is a p-group. Therefore N is a central simple ring.

One might suspect that N is a ring if some $e_i Ne_i$ is a finite additive group, but this is not the case. Professor Wielandt has informed me that the set of all identity-fixing mappings of the simple group A_5 into itself is a simple DGN-ring A, obviously finite. Here each $e_i Ae_i$ contains a single element different from 0, and at least one of these elements must be of order 2, since A_5 is of even order.

REFERENCES

1. D. W. Blackett, *Simple and semisimple near-rings*, Proc. Amer. Math. Soc. 4 (1953), 772-785.

2. W. E. Deskins, *A radical for near-rings*, Proc. Amer. Math. Soc. 5 (1954), 825-827.

3. H. Fitting, *Die Theorie der Automorphismenringe Abelscher Gruppen und ihr Analagon bei nicht kommutativen Gruppen*, Math. Ann. 107 (1932), 514-542.

4. N. Heerema, *Sums of normal endomorphisms*, Trans. Amer. Math. Soc. 84 (1957), 137-143.

5. H. Zassenhaus, *The theory of groups*, Chelsea, New York, 1949.

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Added in proof: In a recent paper, *The near-ring generated by the inner automorphisms of a finite simple group* (J. London Math. Soc. 33 (1958), 95-107), A. Frölich has proved that the set of all identity-fixing mappings of any simple non-abelian group into itself is a simple DGN-ring.