A REAL CONTINUOUS FUNCTION ON A SPACE ADMITTING TWO PERIODIC HOMEOMORPHISMS

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This note is concerned with a map invariant under the even powers of two periodic fixed-point-free homeomorphisms of even periods $2m_1$ and $2m_2$. A special case with $m_1 = m_2 = 1$ is known [3], also a case with $m_1 = m_2 = 2$ [1]. The latter is of interest as the basis of the proof in [1] that the intersection of the boundary of a strictly convex set in Euclidean 3-space and some sphere contains the endpoints of three orthogonal diameters. (Presumably, the boundary of a convex body in E^n contains the endpoints of n mutually orthogonal diameters [of equal lengths] through some point; but the writer has not yet succeeded in proving this.)

A covering space of Y is a pair (P, p), where P is a connected, locally connected Hausdorff space, and where p is a (projection) map of P onto Y such that each $y \in Y$ admits a connected *evenly* covered open neighborhood [2, p. 41]. Such open neighborhoods are called *preferred sets*. By a measure μ we shall mean a monotone set function to the nonnegative reals which is defined for all open and closed subsets, is at least finitely additive on disjunct open or closed sets, and has the property $\infty > \mu(P) \geq \mu(O) > 0$, for each open set O. Write C(2m) for the cyclic group of order 2m, $\{R^n | n = 0, \cdots, 2m - 1\}$. The elements $R^{2\ell}$ ($\ell = 0, 1, 2, \cdots$) are the *even* elements, and the elements $R^{2\ell+1}$ are the *odd* elements of the group.

We shall say that y_1, y_2 in Y are *chained* if there is a finite ordered collection of preferred sets $\{O_i | i = 1, \cdots, n\}$ such that $O_i \cap O_{i+1}$ is not empty and $y_1 \in O_1$, $y_2 \in O_n$. If $\{O_i\} \subset U$, then y_1 and y_2 are *chained in* U.

THEOREM. *Let* P *be a unicoherent locally connected compactum with points* $\{r\}$. *Suppose* P *admits a measure* μ, *and suppose* R_1 *and* R_2 *are two measure-preserving homeomorphisms of minimum periods* $2m_1$ *and* $2m_2$, *respectively, with* R_i^j $(j \leq m_i)$ *fixed-point-free. Let* f *be a continuous real-valued function on* P, *subject to the condition*

$$(\alpha) \qquad f(rR_i^{2\ell_i}) = f(r) \qquad (\ell_i = 1, \cdots, m_i - 1;\ i = 1, 2).$$

Then, for some $\bar{r} \in P$, $f(\bar{r}) = f(\bar{r}R_1) = f(\bar{r}R_2)$.

We follow, in general, the ideas of our proof of the special case where P is projective 3-space [1, Theorem 3]. For each $r \in P$, the transforms $r, rR_i^2, \cdots, rR_i^{2m_i-2}$ are distinct, since, after reduction mod $2m_i$, $0 \leq \min(s_i, m_i, - s_i) \leq m$. Accordingly, let Y_i be the identification space with points

$$y = y(r) = (r, rR_i^2, \cdots, rR_i^{2m_i-2}).$$

Let T_i project P on Y_i by $T_i r = y = y(r)$ $(= y(rR_i^{2\ell_i})$. Write R, Y, T for the corresponding entities either with the subscript 1 or the subscript 2. The topology of Y is defined by taking V as open in Y if and only if $T^{-1}V$ is open in P. By compactness, $\inf d(r, \bigcup_{j=1}^{j=2m-1} rR^j) > 0$, where d(,) is the metric. Hence (P, T),

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or less accurately P, is a covering space. The local connectedness and unicoherence of P imply that Y, also, is locally connected and unicoherent [5]. Let 'R be the fixed-point-free involution on Y defined by

$$y' = y\,'R = (rR, \cdots, rR^{2m-1}).$$

Since $f(rR^{2\ell})$ is independent of ℓ, we may define the continuous function ψ by

$$\psi(y) = f(r), \qquad y = Tr.$$

Let

$$F(y) = \psi(y) - \psi(y\,'R) \qquad (= f(r) - f(rR) \text{ for } y = y(r)).$$

Then $F(yR) = -F(y)$. Except for the trivial case where $f(r)$ is independent of r, the sets

$$'A = \{y \mid F(y) > 0\}, \qquad 'A\,'R = \{y \mid F(y) < 0\}$$

are open and disjunct. The closed set $'C = \{y \mid F(y) = 0\}$ separates 'A and 'A 'R. According to a result of Floyd [3], 'C contains a continuum $'\delta = '\delta\,'R$, devoid of inner points, which is minimal with respect to separating 'A and 'A'R. Write 'W and 'W 'R for the two open disjunct sets satisfying the conditions

$$'A \subset 'W, \qquad 'A\,'R \subset 'W\,'R, \qquad '\overline{W} \cap '\overline{W}\,'R = '\delta.$$

That the sets $'\overline{W}$ and $'\overline{W}\,'R$ are connected follows from the relation $'\overline{W} = 'W \cup '\delta$ [6, p. 20].

Let 'N be a connected, 'R-symmetric (that is, 'N = 'N'R) open set including $'\delta$. We assert that *the set* $A = T^{-1}('W \cup 'N)$ *consists of a single component.* We observe first that any pair of points of a connected set, say 'D, in Y is chained; hence an easy argument based on lifting the chains to P shows that $T^{-1}\,'D$ has at most m components, each of which projects by T onto 'D. Suppose

$$\overline{W} = T^{-1}\,'\overline{W} = \bigcup_{i=1}^{\ell} K_i,$$

where K_i is a component of $\overline{W}$. Then, since $\overline{W} = \overline{W}R^2 = \cdots = \overline{W}^{2m-2}$, there exists, for each i, an index $j = j(i)$ with $K_iR^2 = K_j$. Hence, since $TK_i = '\overline{W}$, ℓ is a divisor of m. Similarly,

$$N = T^{-1}\,'N = \bigcup_{j=1}^{p} G_j,$$

where p is a divisor of m and G_j is a component of N.

Since 'N is connected, each pair of points y_1, y_2 in 'N is chained in 'N. Since $y \in 'N$ implies that $y\,'R \in 'N$, y and y 'R are chained in 'N by $\{'O_i \mid i = 1, \cdots, n\}$. Let r be any point of $T^{-1}y$. The chain of preferred sets can be lifted to the unique chain $\{O_i \mid O_i \in T^{-1}\,'O_i,\ O_1 \supset r\}$. Then $rR^{2s_1+1} = T^{-1}(y\,'R) \cap O_n$. It is thus clear that $G_j = G_j R^{2s_1+1}$, where s_1 is independent of j. In short, there exists a maximal subgroup H of the cyclic group $C(2m) = \{R^j \mid j = 0, \cdots, 2m - 1\}$ which leaves each G_j invariant. Moreover, H contains an odd element, namely R^{2s_1+1}. Let π be a subset of the integers $1, \cdots, \ell$, and write π' for its complement. Similarly, let ρ and ρ' stand for a subset and its complement in $1, \cdots, p$. Write

$$K(\pi) = \bigcup_{i \in \pi} K_i, \qquad G(\rho) = \bigcup_{i \in \rho} G_j.$$

If our assertion is false, then one of the components of A is

$$C = K(\pi) \cup G(\rho),$$

where ρ is a proper subset.

Let J be the maximal subgroup of C(2m) leaving $G(\rho)$ invariant. Since $J \supset H$, it must include an odd element, say R^{2s_1+1}. It is trivial that J actually contains as many odd elements as even elements (for by composition with R^{2s_1+1} or R^{2m-2s_1+1}, each even element $R^{2\ell}$ can be matched with a unique odd element $R^{2\ell+1}$, and each odd element $R^{2\ell+1}$ with $R^{2\ell-2s_1}$, in such a way that distinct elements match distinct elements). The subgroup consisting of the even elements in J is denoted by E. Since $\{R^j\}$ is fixed-point-free, for $r_0 \in K(\pi) \cap G(\rho)$ the even transforms of r_0 in C consist of $\{r_0 R^{2t} \mid R^{2t} \in E\}$, and we denote them by $r_0 E$. The collection of all odd transforms of r_0 which are in $G(\rho)$ is then $\{r_0 R^{2t+2s_1+1} \mid R^{2t} \in E\}$, and we denote it by $r_0 E R^{2s_1+1}$. Observe that the invariance group for $K(\pi)$ is E. Let π_1 be the subset in one-to-one correspondence with π defined by $\pi_1 = \{k \mid K_k = K_j R^{2s_1}, j \in \pi\}$, so that π_1 and π have the same cardinality. Then the transforms of r_0 in $K(\pi_1)R$ are precisely those of $r_0 E R^{2s_1+1}$, so that there are no unshared transforms of r_0 either in $G(\rho)$ or $K(\pi_1)R$. Hence $K(\pi) \cup G(\rho) \cup K(\pi_1)R$ is a proper component of the connected set P, and this is absurd. The italicized assertion is, accordingly, validated. Since $T^{-1}('W'R \cup 'N) = AR$ is a homeomorph of A, it, too, is connected.

Write $\delta = T^{-1}\,'\delta$. We assert that $\overline{W} \cap \overline{WR} = \delta$. Let $r \in \delta$; then each sufficiently small neighborhood O of r is homeomorphic to its projection by T. Thus TO intersects both 'W and 'W'R, whence O must intersect both W and WR; that is, r is a boundary point of both W and WR.

To show that δ is connected, we first remark that $N = T^{-1}\,'N$ is connected. Specifically, the local connectivity, together with the unicoherence of P, guarantees that if the union of two connected *open* sets covers P, their intersection must be connected [4], whence

$$T^{-1}('W \cup 'N) \cap T^{-1}('W'R \cup 'N) = (W \cup N) \cap (WR \cup N) = N$$

is connected.

Next, by the connectedness of $'\delta$,

$$T^{-1}\,'\delta = \delta = \bigcup_{k=1}^{q} \delta(k),$$

where q divides 2m. Since $'\delta$ is 'R-symmetric, it follows as in the case of 'N that there exists a maximal subgroup of C(2m) (we denote it by L) which leaves each $\delta(k)$ invariant, and that this subgroup contains an odd element. By the normality of

P, there are q disjunct open sets $\{M_k \mid k = 1, \cdots, q\}$ in P with $\delta(k) \subset M_k$. Then the L-symmetric set

$$N_k = \bigcap_{R^s \in L} M_k R^s \subset M_k$$

contains $\delta(k)$ also. Define $'N$ as the component of $\bigcap {}'TN_k$ containing $'\delta$. This is an $'R$-symmetric neighborhood of $'\delta$. Accordingly,

$$N\ (= T^{-1}\,'N) = \bigcup (N_k \cap T^{-1}\,'N),$$

which is self-contradictory unless q = 1; for the right-hand side contains q components, whereas the left-hand set is connected as has been shown above. Accordingly, δ is connected.

Thus $\delta_i = T_i^{-1}\,'\delta_i$ (i = 1, 2) yields a separation into disjunct open sets,

$$P - \delta_i = W_i \cup W_i R_i,$$

with

$$\overline{W}_i \cap \overline{W}_i R_i = \delta_i .$$

If $\overline{W}_1 = W_1 \cup \delta_1$ failed to meet $\overline{W}_2$, then $\mu(W_2 R_2 - \overline{W}_1) > 0$. Hence

$$\mu(W_2 R_2) > \mu(\overline{W}_1) .$$

On the other hand, by the R_i-invariance of μ,

$$2\mu(W_2 R_2) = \mu(W_2) + \mu(W_2 R_2) \leq \mu(P) \leq \mu(\overline{W}_1) + \mu(\overline{W}_1 R_1) = 2\mu(\overline{W}_1),$$

a contradiction. Similarly $\overline{W}_1$ meets $\overline{W}_2 R_2$. Accordingly, since $\overline{W}_1 = W_1 \cup \delta_1$, $\overline{W}_1$ is connected [6, p. 20], and so the sets $D_1 = \overline{W}_1 \cap \delta_2$ and $D_2 = \overline{W}_1 R \cap \delta_2$ are not empty. If $D_1 \cap D_2 = \delta_1 \cap \delta_2$ were empty, then D_1, D_2 would be a partition of the connected set δ_2 into two disjunct closed sets, a manifest absurdity. The common points of δ_1 and δ_2 satisfy the assertion of the theorem.

COROLLARY 1. *If* P *is a unicoherent, locally connected compactum, and* R *is a homeomorphism of minimum period* 2m, *where* R *is fixed-point-free, and if* S *is* (a) *a closed set*, (b) *invariant under the cyclic group* $C(2m) = \{R^j \mid j = 0, \cdots, 2m - 1\}$, *where* (c) $r \in P - S$ *is separated by* S *from* rR, *then there exists a minimum continuum* δ, *with empty interior, which is contained in* S *and satisfies* (a), (b) *and* (c).

COROLLARY 2. *In the theorem above, replace* f *by* f_1 *and* f_2, *where*

$$f_i(rR^{2\ell_i}) = f_i(r) \qquad (\ell_i = 1, \cdots, 2m_i = 1) .$$

Then, for some $\bar{r} \in P$,

$$f_1(\bar{r}) = f_1(\bar{r}R_1), \qquad f_2(\bar{r}) = f_2(\bar{r}R_2) .$$

COROLLARY 3. *In the theorem, omit* (α). *Let*

$$F_i(r) = \sum_{j=0}^{j=m_i-1} [f(rR^{2j}) - f(rR^{2j+1})].$$

Then, for some $\bar{r}$, $F_1(\bar{r}) = 0 = F_2(\bar{r})$.

REFERENCES

1. D. G. Bourgin, *Some mapping theorems*, Rend. di Mat. Roma 15 (1956), 177-189.
2. C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, 1946.
3. E. E. Floyd, *Real-valued mappings of spheres*, Proc. Amer. Math. Soc. 6 (1955), 957-959.
4. A. H. Stone, *Incidence relations in unicoherent spaces*, Trans. Amer. Math. Soc. 65 (1949), 427-447.
5. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28 (1942).
6. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1949).

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