HYPERCOMPLETE LINEAR TOPOLOGICAL SPACES

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The classical closed-graph theorem of Banach states that a linear transformation T of a complete metric linear space E into a second space F of the same type is continuous whenever it has a closed graph. More generally, T is continuous provided the inverse of each nonvoid open set in F is somewhere dense, T has a closed graph, E is metrizable, and F is a complete metric space. This theorem does not hold for complete linear topological spaces E and F, as the following example shows. If E is any infinite-dimensional Banach space and F is the same space with the strongest (largest) locally convex Hausdorff topology, then the identity map T has the property that the inverse of each nonvoid open set is somewhere dense (by way of a simple category argument), and T surely has a closed graph since T^{-1} is continuous. Yet T is not continuous, since F is not metrizable. Moreover, it is easily seen that in this case F is complete, *tonnelé*, reflexive and bornological because F is isomorphic to a direct sum of copies of the scalar field. Thus the validity of the closed-graph theorem requires a stronger hypothesis than any of the usual linear topological space specifications.

The purpose of the investigation reported here is to describe the class of locally convex Hausdorff spaces F for which a closed-graph theorem holds for all possible spaces E (a more precise statement is given later). It will be shown that such spaces F are precisely those which satisfy a weakened form of the requirement that the class $\mathscr{C}$ of all convex circled subsets of F be complete relative to the Hausdorff uniformity. We shall call F hypercomplete if $\mathscr{C}$ is complete; a complete metric space is automatically hypercomplete. It will be shown that F is hypercomplete if and only if each convex circled subset A of the adjoint F* is weak*-closed whenever its intersection with each equicontinuous set B is closed in B; that is, hypercompleteness is equivalent to the well-known theorem of Krein and Šmulian.

Thus both the closed-graph theorem and the Krein-Šmulian theorem hold for hypercomplete spaces. These two propositions are perhaps the most striking consequences of a nonmetric completeness requirement that have been attained. Indeed, the general notion of completeness has played a very disappointing role in linear topological space theory in contrast to its basic importance in normed space theory. It now appears that, whereas the class of *tonnelé* spaces is the natural extension of the second category class, hypercomplete spaces are in similar position with respect to complete metric spaces.

Previous work in this direction includes the following. Ptak [5] showed that the open-mapping theorem for F (see Theorem 2) is equivalent to the property (W) that a subset of F* be weak*-closed if its intersection with each equicontinuous subspace B of F* is weak*-closed in B. Collins [2] studied property (W) further, and established some permanence properties. Recently A. P. and W. Robertson [6] proved the closed-graph theorem for E *tonnelé* and F with property (W).

There are several open questions on hypercompleteness. The first of these concerns the permanence properties on which much of the usefulness of the notion depends. It will be shown in what follows that closed subspaces and quotients of

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hypercomplete spaces are hypercomplete. On the other hand, a direct sum of hypercomplete spaces may fail to be hypercomplete, as is shown by the example given earlier. An arbitrary product of hypercomplete spaces may fail to be hypercomplete, because each complete space is topologically isomorphic to a closed subspace of a product of Banach spaces, and not every complete space is hypercomplete. I do not know whether the product of two hypercomplete spaces is of the same sort; I conjecture that each countable product or countable direct sum of hypercomplete spaces is hypercomplete.

Finally, the question of "hypercompletion" arises. We cannot expect to embed topologically an arbitrary linear topological space in a hypercomplete space, since a complete space is necessarily closed and would inherit the hypercompleteness of the containing space. However, one might ask whether, for each space E, it is possible to construct a hypercomplete space $E^\wedge$ containing E, such that each continuous linear map T of E into a space F has an extension $T^\wedge$ carrying $E^\wedge$ into $F^\wedge$, subject to the natural requirements: (i) if T is an identity map, then so is $T^\wedge$, (ii) the map $T \to T^\wedge$ is a one-to-one linear transformation of the space L(E, F) of linear maps of E into F into $L(E^\wedge, F^\wedge)$, and (iii) if S and T are linear transformations, then $(S \circ T) = S^\wedge \circ T^\wedge$.

1. COMPLETENESS PROPERTIES

All spaces considered will be assumed to be locally convex linear topological Hausdorff spaces, with either the real or complex numbers as scalar field. A *local base* for the topology of a space F is a base for the neighborhood system of O; thus local convexity is equivalent to the statement that the class of convex circled neighborhoods of O (that is, the class of convex neighborhoods closed under multiplication by scalars of absolute value at most one) is a local base for the topology. If $\mathscr{U}$ is a local base, then the closure A^- of any set A is a subset of A + U for every U in $\mathscr{U}$, and in fact, $A^- = \bigcap \{A + U: U \in \mathscr{U}\}$.

The family $\mathscr{A}$ of all subsets of a linear topological space F has a natural uniform structure which is described as follows. For each neighborhood U of O in F, let $W_U = \{(A, B): A \subset B + U$ *and* $B \subset A + U\}$. The family of all sets of the form W_U is the base of a uniformity for $\mathscr{A}$ which we shall call the *Hausdorff uniformity*. If F is metric, the Hausdorff uniformity is precisely the uniformity of the Hausdorff metric for $\mathscr{A}$. It is known that if F is metric and complete, then $\mathscr{A}$ is complete; the proof of this fact is elementary, since the set to which a Cauchy sequence $\{A_n\}$ converges is simply the set of all limit points of Cauchy sequences of points $\{x_n\}$ such that $x_n \in A_n$ for each n. However, as will be demonstrated, it is in general impossible to infer completeness of $\mathscr{A}$ from that of F.

We shall be particularly concerned with the completeness of the family $\mathscr{C}$ of all convex circled subsets of F; if $\mathscr{C}$ is complete, relative to the Hausdorff uniformity, then F is called *hypercomplete*. It is possible to describe hypercompleteness rather simply in the following terms. A family $\mathscr{F}$ of nonvoid convex circled subsets of F will be called *fundamental* if and only if (i) it is directed by $\subset$ (that is, if it is a filter base), and (ii) if for each neighborhood U of O in F there is B in $\mathscr{F}$ such that $B \subset A + U$ for every A in $\mathscr{F}$. The fundamental family $\mathscr{F}$ *converges* provided, with the notation $C = \bigcap \{A^-: A \in \mathscr{F}\}$, it is true that for each neighborhood U of O in F the set C + U contains a member of $\mathscr{F}$ (in other words, $\mathscr{F}$ is eventually in C + U).

THEOREM 1. *The linear topological space* F *is hypercomplete if and only if each fundamental family in* F *converges.*

Proof. If F is hypercomplete and $\mathscr{F}$ is a fundamental family in F, then $\mathscr{F}$ is directed by $\subset$, and the net $\{A, A \in \mathscr{F}\}$ is clearly a Cauchy net relative to the Hausdorff uniformity; it therefore converges relative to this uniformity to a member C of $\mathscr{C}$. Consequently, for each neighborhood U of O there exists a member A of $\mathscr{F}$ which is contained in C + U and hence in $C^- + U$. The fundamental family $\mathscr{F}$ will then be shown to converge if it is proved that $C^- = \bigcap\{A^-: A \in \mathscr{F}\}$. For each neighborhood U of O, for some A in $\mathscr{F}$, $C + U \supset A$, whence

$$C^- + U + U \supset A + U \supset A^- \supset \bigcap\{B^-: B \in \mathscr{F}\},$$

and hence C^- contains the intersection of the closures of the members of $\mathscr{F}$. On the other hand, for each U it is true that A + U eventually (and hence always) contains C, and since this inclusion holds for fixed A and all neighborhoods U of O, it follows that $A^- \supset C$, whence the intersection $\bigcap\{A^-: A \in \mathscr{F}\}$ contains C^-. Thus equality is proved.

Conversely, if $\{A_n, n \in D\}$ is an arbitrary Cauchy net in $\mathscr{C}$ and C_m is the convex extension of $\bigcup\{A_n: n \geq m\}$, then the family of all sets of the form C_m is easily seen to be fundamental. If this family converges, $C = \bigcap\{C_m^-: m \in D\}$, and U is a neighborhood of O, then C + U contains C_m^- and hence contains C. It follows that $\{A_n, n \in D\}$ converges to C, relative to the Hausdorff uniformity.

A weakening of the notion of hypercomplete will be useful. A fundamental family $\mathscr{F}$ in F will be called *scalar* provided that if $A \in \mathscr{F}$ and r is a positive real number, then $rA \in \mathscr{F}$; and F will be called *fully complete* provided each scalar fundamental family converges. (H. Collins has defined "fully complete" in a way which one of the theorems of this paper shows to be equivalent to the preceding.) It is clear that if $\mathscr{F}$ is a scalar fundamental family, then $\bigcap\{A^-: A \in \mathscr{F}\}$ is a closed subspace of F. By using the argument of the proof of the preceding proposition, one can see that if F is fully complete, then the class of all closed subspaces of F is complete relative to the Hausdorff uniformity; the converse does not hold. I do not know whether a fully complete space must be hypercomplete.

Certain elementary properties of fundamental families will be useful.

LEMMA 1. *If* T *is a continuous linear topological space* E *and* $\mathscr{F}$ *is a fundamental family in* F, *then the class* $\{T[A]: A \in \mathscr{F}\}$ *of images of members of* $\mathscr{F}$ *is fundamental in* E. *Moreover, if* $\mathscr{F}$ *converges, so does* $\{T[A]: A \in \mathscr{F}\}$.

If T *is a continuous open linear map of* F *onto a linear topological space* E, *and if* $\mathscr{G}$ *is a fundamental family in* E, *then* $\{T^{-1}[B]: B \in \mathscr{G}\}$ *is fundamental in* F.

Proof. It is straightforward to verify that the images of the members of a fundamental family form a fundamental family, and that convergence is preserved. If T is continuous and open, $\mathscr{G}$ is a fundamental family in E, and U is a neighborhood of O in F, then T[U] is a neighborhood of O in E, and hence for some B in $\mathscr{G}$ it is true that $T[U] + B \supset A$ for all A in $\mathscr{G}$. A small calculation then shows that

$$U + T^{-1}[B] \supset T^{-1}[T[U] + B] \supset T^{-1}[A],$$

and consequently $\{T^{-1}[B]: B \in \mathscr{G}\}$ is fundamental.

It is an immediate consequence of the lemma that if F is a hypercomplete space and N is a closed subspace, then F/N, with the quotient topology, is hypercomplete; for if $\mathscr{F}$ is fundamental in F/N, the class of inverse images of members of $\mathscr{F}$ under the quotient map is fundamental in F, hence converges, and consequently the class of images (that is, $\mathscr{F}$) converges. The same argument shows that a quotient space of a fully complete space is fully complete, since the class of all images and the class of all inverses of the members of a scalar family are scalar families. It is also clear that closed subspaces of spaces possessing either completeness property possess the same property. Thus:

COROLLARY 1. *If* F *is a linear topological space which is hypercomplete (respectively, fully complete) then each closed subspace of* F *and each quotient space is hypercomplete (respectively, fully complete).*

Remark. It is rather easy to see that if F is hypercomplete, then it is complete, for if $\{x_n, n \in D\}$ is a Cauchy net in F, then $\{C_n, n \in D\}$, where $C_n = \{ax_n: |a| \leq 1\}$ is a Cauchy net in $\mathscr{C}$. It is true that if F is fully complete, then F is complete, but this fact apparently lies a little deeper; it is a consequence of one of our principal results.

2. THE CLOSED-GRAPH AND OPEN-MAPPING THEOREMS

Let T be a linear transformation of a linear topological space E into a space F. We shall say that T^{-1} is *somewhere dense* if and only if for each neighborhood U of O in F it is true that $(T^{-1}[U])^-$ is a neighborhood of O in E. Note that T^{-1} is always somewhere dense if E is of the second category, (more generally, if E is *tonnelê*). The general problem of this section is: For what spaces F is it true that each linear transformation T of an arbitrary linear topological space E into F is continuous provided T has a closed graph and T^{-1} is somewhere dense? Such spaces F are said to have the *closed-graph property.* A classical theorem of Banach asserts that complete metrizable linear spaces possess the closed-graph property.

There is an open-mapping theorem which, in a sense, is dual to the closed-graph theorem. Let us call a transformation T *somewhere dense* if and only if for each neighborhood U of O it is true that $(T[U])^-$ is a neighborhood of O in the range space. As above, if T maps F *onto* E, and E is of the second category (or, more generally, *tonnelê*), then T is automatically somewhere dense. We shall seek conditions under which a somewhere dense transformation of a space F is necessarily open.

The connection between the notation of fundamental family and that of a transformation whose inverse is somewhere dense will now be established. Suppose that T is a linear transformation of E into F, that $\mathscr{U}$ is a local base for the topology of E, and that $\mathscr{V}$ is a local base for the topology of F. Then:

LEMMA 2. *The family* $\{T[U]: U \in \mathscr{U}\}$ *is a fundamental family if and only if* T^{-1} *is somewhere dense.*

Proof. By definition, T^{-1} is somewhere dense if and only if for each V in $\mathscr{V}$ there is U in $\mathscr{U}$ such that $U \subset T^{-1}[V] + W$ for all W in $\mathscr{U}$ (this is equivalent to $U \subset T^{-1}[V]^-$). Clearly this implies that $T[U] \subset V + T[W]$ for all W; that is, that $\{T[U]: U \in \mathscr{U}\}$ is fundamental. Conversely, if $\{T[U]: U \in \mathscr{U}\}$ is fundamental, then for each V in $\mathscr{V}$ there exists U in $\mathscr{U}$ such that $T[U] \subset V + T[W]$ for all W in $\mathscr{U}$, whence

$$U \subset T^{-1}[T[U]] \subset T^{-1}[V + T[W]] \subset T^{-1}[V] + W,$$

and therefore T^{-1} is somewhere dense.

The next lemma gives a useful characterization of transformations having a closed graph.

LEMMA 3. *Let* T *be a linear transformation of a linear topological space* E *into a linear topological space* F, *let* $\mathscr{U}$ *be a local base for the topology of* E, *let* $\mathscr{V}$ *be a local base for the topology of* F, *and let* $\mathscr{W}$ *be the family of all sets of the form* $T[U]+V$ *for* U *in* $\mathscr{U}$ *and* V *in* $\mathscr{V}$. *Then the graph of* T *is closed if and only if*

$$\bigcap \{T[U]+V\colon U \in \mathscr{U} \text{ and } V \in \mathscr{V}\} = \{0\},$$

or equivalently, if and only if the topology with local base $\mathscr{W}$ *is a Hausdorff topology.*

Proof. The graph of T is closed if and only if for all (x, y) in $E \times F$ with $y \neq T(x)$, there exists U in $\mathscr{U}$ and V in $\mathscr{V}$ such that $(x+U) \times (y+V)$ is disjoint from the graph of T. Translation of $E \times F$ by $(-x, -T(x))$ is a linear homeomorphism leaving the graph of T invariant, and it follows that the graph of T is closed if and only if for each nonzero element z $(=y - T(x))$ of F there exist U in $\mathscr{U}$ and V in $\mathscr{V}$ such that $U \times (z+V)$ is disjoint from the graph of T, or equivalently, such that $z \notin T[U] - V$. Thus the graph of T is closed if and only if

$$\{0\} = \bigcap \{T[U] - V\colon U \in \mathscr{U} \text{ and } V \in \mathscr{V}\} = \bigcap \{T[U] + V\colon U \in \mathscr{U} \text{ and } V \in \mathscr{V}\},$$

and this is simply the condition that the topology with local base $\mathscr{W}$ be a Hausdorff topology.

Our special concern will be spaces F such that the closed-graph property holds for all spaces E and all quotient spaces of F. The reason for this is that an open-mapping theorem holds for such spaces F, as noted in the following lemma on open mappings.

LEMMA 4. *Let* S *be a linear map of a space* F *onto a space* E, *let* N *be the null space of* S, *and let* T *be the induced map of* F/N *onto* E. *Then* T^{-1} *has a closed graph if and only if* S *has a closed graph; moreover,* T *is somewhere dense if and only if* S *is somewhere dense; and* S *is open if and only if* T^{-1} *is continuous.*

In particular, if S *is somewhere dense and has a closed graph, then* S *is open provided* (E, F/N) *has the closed-graph property.*

Proof. Let $\mathscr{U}$ be a local base for E, $\mathscr{V}$ a local base for F, and Q the quotient map of F onto F/N. The family of all Q[V], for V in $\mathscr{V}$, is a local base for the quotient topology for F/N. In view of Lemma 3, T^{-1} has a closed graph if and only if the topology with local base consisting of all sets of the form $T^{-1}[U] + Q[V]$, with U in $\mathscr{U}$ and V in $\mathscr{V}$, is a Hausdorff topology. From the definition of Q, this is the case if and only if for each member y of F with $y \notin N$, there exist U in $\mathscr{U}$ and V in $\mathscr{V}$ such that $y \notin S^{-1}[U] + V + N$. Because S maps F onto E, this is equivalent to the following requirement: if x $(= S(y))$ is a nonzero element of E, then $x \notin U + S[V]$ for some U in $\mathscr{U}$ and some V in $\mathscr{V}$. But, again by Lemma 3, this is true precisely if and only if S has a closed graph. Hence T^{-1} has a closed graph if and only if S has a closed graph. Finally, it is evident from the definition of the quotient topology that T is somewhere dense if and only if S is somewhere dense, and it is well known that S is open if and only if T is open, which is the case if and only if T^{-1} is continuous.

The following is the principal result of the section.

THEOREM 2. *The following statements about a linear topological space* F *are equivalent:*

(i) *The space* F *is fully complete.*

(ii) *Any linear map* T *into a quotient space of* F *is continuous provided* T *has a closed graph and* T^{-1} *is somewhere dense.*

(iii) *Each somewhere dense, linear transformation with closed graph of* F *onto a linear topological space is open.*

(iv) *Each continuous, somewhere dense, linear transformation of* F *onto a linear topological space is open.*

Proof. The pattern of proof is (i) $\to$ (ii) $\to$ (iii) $\to$ (iv) $\to$ (i). The implication (ii) $\to$ (iii) is established by Lemma 4, and (iii) $\to$ (iv) is obvious; therefore only the first and last of the sequence require proof. To prove that (i) $\to$ (ii), assume that N is a closed subspace of a fully complete space F, that Q is the quotient map of F onto F/N, and that T is a linear map of a space E into F/N such that T^{-1} is somewhere dense and the graph of T is closed. If $\mathscr{U}$ is the class of convex circled neighborhoods of O in E, we know from Lemma 2 that the family $\{T[U]: U \in \mathscr{U}\}$ is fundamental in F/N. In view of Lemma 1, the family $\{Q^{-1}[T[U]]: U \in \mathscr{U}\}$ is fundamental in F, and since F is assumed to be fully complete, this family converges. But the continuous image of a convergent fundamental family is also convergent, and hence $\{T[U]: U \in \mathscr{U}\}$ converges. Thus, for each neighborhood V of O in F/N there exists U in $\mathscr{U}$ such that $T[U] \subset V + \bigcap\{(T[S])^-: S \in \mathscr{U}\}$. But, if $\mathscr{W}$ is the class of neighborhoods of O in F/N, then

$$T([S])^- = \bigcap\{T[S] + W: W \in \mathscr{W}\},$$

hence

$$\bigcap\{(T[S])^-: S \in \mathscr{U}\} = \bigcap\{T[S] + W: S \in \mathscr{U} \text{ and } W \in \mathscr{W}\},$$

and since the graph of T is closed, this last intersection is $\{O\}$, by Lemma 3. Thus $T[U] \subset V + \{O\}$, and continuity is proved.

It remains to prove that (iv) $\to$ (i). Suppose that $\mathscr{F}$ is a scalar fundamental family in F, and that $N = \bigcap\{A^-: A \in \mathscr{F}\}$. Then N is a closed subspace of F, and we denote by Q the quotient map of F onto F/N. We construct a topology for F/N as follows. Let $\mathscr{U}$ be the class of all convex circled neighborhoods of O in F, let $\mathscr{V}$ be the class $\{Q[A + U]: A \in \mathscr{F} \text{ and } U \in \mathscr{U}\}$, and let F/N have the topology with local base $\mathscr{V}$. Evidently the map Q is continuous relative to this topology, and we note that the topology is a Hausdorff topology, since

$$\bigcap\{A + U + N: A \in \mathscr{F} \text{ and } U \in \mathscr{U}\} = \bigcap\{A + U: A \in \mathscr{F} \text{ and } U \in \mathscr{U}\}$$
$$= \bigcap\{A^-: A \in \mathscr{F}\} = N.$$

If it is shown that Q is open, then it will follow that $\mathscr{F}$ converges: if for each U in $\mathscr{U}$ there exist V in $\mathscr{V}$ and A in $\mathscr{F}$ such that $Q[U] \supset Q[A + V]$, then, taking inverse images under Q, we have $U + N \supset A + V + N \supset A$. In view of the hypothesis, the proof then reduces to showing that Q is somewhere dense. Since $\mathscr{F}$ is fundamental, for each U in $\mathscr{U}$ there exists A in $\mathscr{F}$ such that $U + B \supset A$ for all B in $\mathscr{F}$; hence $Q[U] + Q[B] \supset Q[A]$, and therefore $Q[U] + Q[B + V] \supset Q[A]$ for all B in $\mathscr{F}$ and V in $\mathscr{V}$. It follows that $(Q[U])^- \supset Q[A]$, whence $(Q[2U])^- \supset (Q[U])^- + Q[U] \supset Q[A + U]$, and it is demonstrated that the closure of the image of each neighborhood of O in F is a neighborhood of O in F/N.

It is interesting to note that the theorem above yields a new proof of the classical closed-graph and open-mapping theorems. The sequential character of the argument becomes apparent in the proof that the space of closed subsets of a complete metric space is complete, relative to the Hausdorff metric; given this, all else is free of countability assumptions.

A few general observations about the preceding theorem should be made. Speaking with some imprecision: a closed-graph theorem for normed domain spaces E would really be adequate for most instances, since from this one can deduce a closed-graph theorem for sequentially complete bornological spaces. In order that an open-mapping theorem hold in addition, we should require the space F to have the property (A): *the closed-graph theorem holds for each normed* E *and each quotient space of* F *by a closed subspace.* I do not know whether (A) implies full completeness or hypercompleteness. By the methods of the preceding sections, one may see that F possesses (A) if and only if, whenever B is a convex circled subset of F such that the scalar multiples of B form a fundamental family (this requirement is equivalent to $eB + U$ absorbing B for each neighborhood U of O and each $e > 0$), then the image of B in the quotient space $F/\bigcap\{eB^-: e > 0\}$ is bounded. Anticipating the methods of the next section, we point out that F has property (A) if and only if, whenever C is a convex circled w*-closed subset of F* such that the linear space G generated by C is ew*-closed, then G is necessarily w*-closed. The Robertsons [6] have shown that the countable inductive limit of fully complete spaces has a weakened closed-graph property which is approximately (A); their result implies the closed-graph theorem which is given in Grothendieck's thesis.

3. DUALIZATION: THE KREIN-ŠMULIAN THEOREM

Let F be a linear topological space, and let F*, its adjoint, be the space of continuous linear functionals on F. The weak*-topology for F* is the topology of pointwise convergence on F. For each subset A of F the *polar* A^o of A is defined to be

$$\{f\colon f \in F^* \text{ and } f(x) \leq 1 \text{ for all } x \text{ in } A\},$$

and, symmetrically, if $B \subset F^*$ then the polar B_o of B is defined to be

$$\{x\colon x \in F \text{ and } |f(x)| \leq 1 \text{ for all } f \text{ in } B\}.$$

If $A \subset F$, then $A^o = (A^-)^o$, ${A^o}_o$ is the closure of the convex circled extension of A, and if $B \subset F^*$ then ${B^o}_o$ is the weak*-closure of the convex circled extension of B. A linear subspace G of F* is weak*-dense in E* if and only if G distinguishes points of F; that is, if for each nonzero member x of F there is a member f of G such that $f(x) \neq 0$. (For these and related facts, see [1] or [4].)

We shall also need a few facts about a linear transformation and its adjoint. If T is a linear transformation of a space E into a space F, then the adjoint T* of T is defined by $T^*(f) = f \circ T$ for all f in F* such that $f \circ T$ is continuous. If the domain of T* is all of F*, then T is necessarily continuous, provided E is a Mackey space (see [1] or [4]). The map T* is always continuous relative to the weak*-topologies.

The ew*-topology for the adjoint F* of a space F is defined by agreeing that a subset B of F* is ew*-closed if and only if $B \cap D$ is weak*-closed in D for each equicontinuous set D. The relation of the ew*-topology to the closed-graph and open-mapping theorems will now be exhibited. The most important implications of the following proposition will also follow from the later theorem which is the major result of the section; however, the arguments given below are of some intrinsic interest.

THEOREM 3. *Let* T *be a linear map of a linear topological space* E *into a space* F, *and let* G *be the class of all* f *in* F* *such that* $f \circ T$ *is continuous. Then*

(i) G *is* ew*-*closed if* T^{-1} *is somewhere dense;*

(ii) *if* E *is a Mackey space and* T^{-1} *is somewhere dense, then* G *is* ew*-*closed; and*

(iii) T *has a closed graph if and only if* G *is weak*-dense in* F*.

In the dual situation: if T *is a somewhere dense, continuous, linear transformation of* F *onto a space* E, *then the image* T*[E*] *is* ew*-*closed in* F*.

Proof. If U is a neighborhood of O in F and $f \in G \cap U^o$, then $f \circ T \in T^{-1}[U]^o$, and since $f \circ T$ is continuous, $f \circ T \in T^{-1}[U]^{-o}$. Consequently, if $(T^{-1}[U])^-$ is a neighborhood of O in E, then $T^*[G \cap U^o]$ is an equicontinuous subset of E*. It follows that for each weak* accumulation point g of $G \cap U^o$ it is true that $g \circ T$ is continuous, hence G is ew*-closed whenever T^{-1} is somewhere dense, and (i) is established. If G is ew*-closed, then, for each convex circled neighborhood U of O in F, the set $G \cap U^o$ is weak*-compact and convex, and $T^*[G \cap U^o]$ is therefore equicontinuous if E is a Mackey space. But, by an elementary calculation, $T^*[G \cap U^o] = T^{-1}[U]^o$, whence $T^{-1}[U]^o{}_o = (T^{-1}[U])^-$ is a neighborhood of O. This establishes (ii).

To establish (iii): By a straightforward verification, we see that G is precisely the set of linear functionals that are continuous with respect to the topology for F which has a local base consisting of all sets $T[U] + V$, where U is an arbitrary neighborhood of O in E, and V an arbitrary neighborhood of O in F. Hence, by the Hahn-Banach theorem, G distinguishes points of F (equivalently, G is weak*-dense in F*) if and only if this is a Hausdorff topology, and according to Lemma 3 this is the case if and only if the graph of T is closed.

The last statement of the theorem is an immediate consequence of the fact that if U is a neighborhood of O in F, then $T^*[E^*] \cap U^o = T^*[T[U]^o]$. This identity may be verified directly. Using it, we see that if T is somewhere dense, then $T[U]^o = [T[U]^-]^o$ is equicontinuous, hence weak*-compact, thus $T^*[T[U^o]$ is weak*-compact, and $T^*[E] \cap U^o$ is weak*-closed. Therefore T*[E*] is ew*-closed.

It is clear from the foregoing that if F is a space such that each ew*-closed subspace is w*-closed (that is, fully complete in the sense of H. Collins [2]), if E is *tonnelé* and T has a closed graph, then T is continuous. This result was proved by the Robertsons [6]; a somewhat stronger proposition of this sort will be obtained here.

The proof of the main theorem of the section requires a little preliminary calculation on polars. For convenience, let us define the set $A \triangle B$, corresponding to subsets A and B of a linear topological space, as the set of all x such that $x \in rA \cap sB$ for some nonnegative real numbers r and s with $r + s \leq 1$. In other words,

$$A \triangle B = \bigcup \{rA \cap sB: r \geq 0,\ s \geq 0, \text{ and } r + s \leq 1\}.$$

If A and B are circled sets, then it is easy to see intuitively the nature of the set $A \triangle B$: the set $A \triangle B$ contains rays only in "directions" common to A and B, and if $mx \in A$ and $nx \in B$, where m and n are positive real numbers, then a simple calculation shows that $[1/(1/m + 1/n)]\, x \in A \triangle B$. Thus, speaking roughly, the set $A \triangle B$

extends in each direction to one half the harmonic mean of the distances to which A and B extend. The most important properties of $\triangle$, and indeed the reason for the definition of the operation, are given in the following lemma.

LEMMA 5. *If* A *and* B *are nonvoid circled subsets of* F, *then* $(A + B)^o = A^o \triangle B^o$.

It is always true that $(1/2)(A \cap B) \subset A \triangle B$, *and if* A *and* B *are circled, then* $A \triangle B \subset A \cap B$.

If $\mathscr{U}$ *is a family of subsets of* F, *then* $A \triangle \bigcup\{B: B \in \mathscr{B}\} = \bigcup\{A \triangle B: B \in \mathscr{B}\}$.

Proof. If $f \in (A + B)^o$, $a = \sup\{|f(x)|: x \in A\}$, and $b = \sup\{|f(x)|: x \in B\}$, then we see that $a + b \leq 1$. Then $f \in aA^o \cap bB^o \subset A^o \triangle B^o$. On the other hand, if $f \in A^o \triangle B^o$, then $f \in rA^o \cap sB^o$ for some nonnegative numbers r and s with sum at most one. Then $|f|$ is at most r on A and at most s on B, and is therefore at most $r + s$ on $A + B$; that is, $|f(x)| \leq r + s \leq 1$ for x in $A + B$, and it is proved that $f \in (A + B)^o$. The last two statements of the lemma are established in a straightforward way, and the proofs are omitted.

We are now in a position to give dualized equivalences to the notions of fully complete and hypercomplete.

THEOREM 4. *A linear topological space* F *is fully complete if and only if each* ew*-*closed subspace of the adjoint space is weak*-closed.*

The space F *is hypercomplete if and only if each* ew*-*closed convex circled subset of the adjoint space is* w*-*closed.*

Proof. We shall give the proof of the second statement only. A proof of the first may be obtained by replacing "fundamental family" everywhere by "scalar fundamental family," and "convex circled subset" by "subspace."

Suppose that F has the property that each ew*-closed convex subset of F* is w*-closed, and that $\mathscr{F}$ is a fundamental family in F. If $\mathscr{G}$ is the class of polars of members of $\mathscr{F}$, then the fact that $\mathscr{F}$ is fundamental implies, by Lemma 5, that for each neighborhood U of O in F there is a member A_U of $\mathscr{F}$ such that $U^o \triangle G \subset (A_U)^o$ for all G in $\mathscr{G}$. Let $P = \bigcup\{G: G \in \mathscr{G}\}$; then $U^o \triangle P \subset (A_U)^o$, and hence $[U^o \triangle P]^- \subset P$, where $^-$ denotes w*-closure. For each neighborhood U of O and each positive n, we then have

$$P \supset [P \triangle nU^o]^- \supset \left[\left(1 - \frac{1}{n}\right)P \cap \left(1 - \frac{1}{n}\right)U^o\right]^-$$

(the last inclusion is easily verified), and hence $U^o \cap P \supset r[U^o \cap P]^-$ for each r $(0 \leq r < 1)$. It follows that if $x \in [U^o \cap P]^-$, then the half-open real line segment $[0, x)$ is a subset of $U^o \cap P$, and hence the set $P' = P \cup \{x: [0: x) \subset P\}$ has the property that $U^o \cap P' \supset [U^o \cap P']^-$. Thus P' is ew*-closed, and consequently, under the assumed hypothesis, P' is w*-closed. Finally, from the fact that for each U there exists A_U in $\mathscr{F}$ such that $U^o \triangle P \subset A_U{}^o$, we deduce that $U^o \triangle P' \subset A_U{}^o$ because $A_U{}^o$ is closed, and hence $U^o{}_o + P'{}_o \supset A_U$ by Lemma 5. Since $P_o = P'{}_o$, we have

$$A_U \subset U^o{}_o + P_o = U^o{}_o + \bigcap\{G_o: G \in \mathscr{G}\} = U^o{}_o + \bigcap\{A^-: A \in \mathscr{F}\}.$$

Thus $\mathscr{F}$ converges.

To prove the converse, suppose that F is hypercomplete, and that A is a convex, circled e-w*-closed subset of F*. Let $\mathscr{F}$ be the family of all sets of the form

$(U^{\circ} \cap A)_o$, where U is a neighborhood of O in F. Clearly $\mathscr{F}$ is directed by $\subset$, and the following argument shows that $\mathscr{F}$ is fundamental. If U and V are neighborhoods of O in F, then

$$(V + (U^{\circ} \cap A))^{\circ} = V^{\circ} \triangle (U^{\circ} \cap A) \subset V^{\circ} \cap A,$$

in view of Lemma 5, and hence

$$V + V + (U^{\circ} \cap A)_o \supset (V + (U^{\circ} \cap A)_o)^{\circ}{}_o \supset (V^{\circ} \cap A)_o.$$

Thus, for each neighborhood W of O there exists a member of $\mathscr{F}$ (namely, $(V^{\circ} \cap A)_o$, where V is any neighborhood such that $V + V \subset W$) which is contained in $W + C$ for each C in $\mathscr{F}$. Since $\mathscr{F}$ is fundamental and F is supposed to by hypercomplete, $\mathscr{F}$ converges. Letting

$$B = \bigcap \{ (U^{\circ} \cap A)_o : U \text{ a neighborhood of O in F} \},$$

we have: for each neighborhood V of O there is U such that $B + V \supset (U^{\circ} \cap A)_o$. Hence, in view of Lemma 5, $B^{\circ} \triangle V^{\circ} \subset U^{\circ} \cap A \subset A$. It follows from this inclusion that if $x \in B^{\circ}$, then the half-open real line segment $[0: x)$ is contained in A, since the union of $B^{\circ} \triangle V^{\circ}$ for all neighborhoods V is $\{x: [0: x) \subset B^{\circ}\}$. But since the set A is ew*-closed, it surely contains x whenever $[0: x) \subset A$, and hence $B^{\circ} \subset A$. On the other hand, $B^{\circ} \supset V^{\circ} \cap A$ for all V, since the polar of $V^{\circ} \cap A$ contains B, and therefore $A \subset B^{\circ}$. Thus $B^{\circ} = A$, consequently A is weak*-closed and the proof is complete.

Presumably the requirement "circled" could be omitted from the statement of the preceding theorem; this would yield a precise generalization of the Krein-Šmulian theorem, but I do not see how to establish this stronger result.

4. THE ew*- AND cew*-TOPOLOGIES

The usual proof of the Krein-Šmulian theorem depends on identifying the topology ew* with the topology of uniform convergence on totally bounded sets. In the following paragraphs we give a few results obtained by applying this sort of argument to the general case. The argument yields some information on the nature of the topology ew*; in particular, it will be seen that ew* is not always locally convex.

The topology cew* for the adjoint F* of a space F is defined to be the strongest locally convex topology which is weaker than ew*. Thus the family of all convex circled ew*-neighborhoods of O is a local base for cew*, and cew* = ew* if and only if ew* is locally convex. Clearly, each linear functional on F* which is cew*-continuous is ew*-continuous. The converse proposition is also true, for if ϕ is ew*-continuous, then $\{f: |\phi(f)| \leq 1\}$ is convex, and is in fact a cew*-neighborhood of O on which ϕ is bounded. Thus F*, with either the ew*- or the cew*-topology, has the same adjoint.

There is another topology for F* which yields the same adjoint. According to a theorem of Grothendieck, the completion $F^{\wedge}$ of F is topologically isomorphic to the class of all linear functions ϕ which are w*-continuous on each equicontinuous subset of F*. But ϕ is w*-continuous on each equicontinuous subset if and only if $\phi^{-1}[O]$ is ew*-closed, which is the case if and only if ϕ is ew*-continuous. Thus ϕ is ew*-continuous on F* if and only if ϕ is evaluation at some point of the completion $F^{\wedge}$ of F. Stated a little more precisely, each member f of F* has a unique continuous extension $f^\wedge$ whose domain is $F^\wedge$, and a linear functional ϕ is ew*-continuous if and only if, for some member x of $F^\wedge$, $\phi(f) = f^\wedge(x)$ for all f in F^*. Thus, if we let $w^\wedge$ denote the weakest topology for F^* such that the map $f \to f^\wedge(x)$ is continuous for each x in $F^\wedge$, a linear functional ϕ is ew*-continuous if and only if it is $w^\wedge$-continuous. The topology $w^\wedge$ will be called the *weak*-topology defined by the completion of* F; F is complete if and only if $w^* = w^\wedge$.

Summarizing the preceding remarks, we have

THEOREM 5. *A linear functional ϕ on F^* is* ew*-*continuous if and only if it is* cew*-*continuous, and this is the case if and only if ϕ is continuous relative to the weak*-topology* $w^\wedge$ *defined by the completion of* F.

If two locally convex topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a linear space E yield the same adjoint E^*, then a convex subset A is $\mathcal{T}_1$-closed if and only if it is $\mathcal{T}_2$-closed, since closure of A is equivalent to weak closure, which is defined in terms of E^*. Hence the

COROLLARY. *A convex subset of* F^* *is* cew*-*closed if and only if it is* $w^\wedge$-*closed.*

Consequently, if ew* *is locally convex and* F *is complete, then each* ew*-*closed convex subset of* F^* *is* w*-*closed.*

Since there exist complete spaces which are not hypercomplete, it follows that the topology ew* is not always locally convex.

The topology cew* has an interesting alternative description. Let $\mathcal{T}_P$ be the topology of uniform convergence on totally bounded *(precompact)* subsets of F. Then

THEOREM 6. *The topology* cew* *for* F^* *is stronger than* $\mathcal{T}_P$, *and if* F *is complete, then* cew* = $\mathcal{T}_P$.

Proof. We first show that if A is a totally bounded subset of F, then A° is a cew*-neighborhood of O, so that cew* is stronger than $\mathcal{T}_P$. If A is totally bounded, then for each neighborhood U of O in F there exists a finite set B such that $A \subset B + (1/2)U$; taking polars, we see by Lemma 5 that

$$A^\circ \supset B^\circ \triangle [(1/2)U]^\circ \supset (1/2)B^\circ \cap (1/2)[(1/2)U]^\circ = (2B)^\circ \cap U^\circ .$$

Hence A° contains the intersection of U° and a w*-neighborhood of O, and it follows that A° is a cew*-neighborhood of O.

If F is complete then, by Theorem 5, each cew*-continuous linear functional on F^* is evaluation at a point of F. Consequently each cew*-closed convex set is w*-closed, and in particular each cew*-neighborhood of O contains a w*-closed, convex, circled cew*-neighborhood of O. If V is such a neighborhood, then ${V_o}^\circ = V$, and if it is shown that V_o is totally bounded, then it will follow that $\mathcal{T}_P$ = cew*. If U is a neighborhood of O in F, then there exists a finite subset B of F such that $V \supset U^\circ \cap A^\circ$, because V is a cew*-neighborhood of O. Hence $V_o \subset \langle U \cup A \rangle^-$, where $\langle\,\rangle$ denotes convex extension. But we may suppose that U is closed and convex, and in this case $\langle U \cup \langle A \rangle\rangle$ is closed, since it is the convex extension of the union of a compact convex set and a closed convex set. Hence $V_o \subset \langle U \cup \langle A \rangle\rangle \subset U + \langle A \rangle$. Since $\langle A \rangle$ is compact, it is totally bounded, and there is therefore a finite set B such that $\langle A \rangle \subset B + U$. Thus $V_o \subset B + U + U$, and it follows that V_o is totally bounded.

We note that w* and $w^\wedge$, the weak*-topology defined by the completion of F, have the same relativization to any $w^\wedge$-compact subset B of F^*, since $w^\wedge$ is stronger than w* and w* is a Hausdorff topology. In particular, this is the case if B is the polar of a neighborhood of O. It follows that, speaking inexactly, the ew*-topology is the same regardless of whether F* is considered as the adjoint of F or as the adjoint of the completion F^. Thus

COROLLARY. *The topology* cew* *for* F* *is the topology of uniform convergence on totally bounded subsets of the completion* F^ *of* F.

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