EQUICONTINUITY AND COMPACTNESS IN LOCALLY CONVEX TOPOLOGICAL LINEAR SPACES

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1. INTRODUCTION

If E and F are locally convex spaces, and L(E, F) is the space of continuous linear mappings of E into F, the equicontinuous subsets of L(E, F) are of natural interest. Indeed, whether or not E is a t-space (*espace tonnelé* [3]) can be stated in terms of a property of such subsets. In this paper, the duality theory of linear spaces is applied systematically, by means of Lemma 2 below, to obtain characterizations of equicontinuity in L(E, F), in several cases in which E and F are given topologies different from the 'Mackey strong' topology τ. In particular, for the case of the topology k (Section 2), there is a natural connection between equicontinuity in L(E, F) and compactness in L(E, F) suitably topologized; the connection becomes especially simple if the spaces E and F satisfy certain restrictions in their τ-topologies. Sections 3, 4, and 5 all bear on the application of the theory in Section 6, where the compact subsets of the algebra of bounded operators on a Hilbert space are characterized in terms of equicontinuity, for several of the topologies studied by Dixmier [8]. Section 4 is devoted to a multiplicative property of equicontinuous subsets of L(E, E). The topological theorem of Section 5 is given more fully than its application to Section 6 requires, because of its possible intrinsic interest.

The symbol □ will denote the end of a proof or of some other expository unit, when paragraphing alone seems insufficient.

2. PRELIMINARIES

A pair of vector spaces E and E', over the same scalar (real or complex) field, are *in duality* if each is a separating set of linear functionals defined on the other. The value of a functional $x' \in E'$ at the point $x \in E$ will be denoted by (x', x). Everything to follow will be quite symmetric as between E and E'; it will therefore suffice to present all definitions and assertions in a one-sided way, the implication of a corresponding dual definition or assertion being understood.

Let θ denote the zero element of E. A topology on E will be named u if u is the set of all neighborhoods of θ, that is, the set of all sets having θ as an interior point. The topology u is *compatible* with the duality of E and E' if it is a locally convex topology on E for which E' is precisely the set of continuous linear functionals. E_u will then denote this locally convex space. If $A \subset E$, we say that $A^0 = \{x' \in E' \mid |(x', x)| \leq 1 \text{ for all } x \in A\}$ (see [3; 4; 5; 7] for such properties of this and other notions herein introduced and used without explitic reference). The weakest compatible topology on E, denoted by $\sigma(E, E')$ or simply by σ, has for a basis (at θ) the collection

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$$\{(A')^0 \mid A' \text{ a finite subset of } E'\}.$$

The strongest compatible topology on E, denoted $\tau(E, E')$ or simply τ, has for a basis the collection

$$\{(A')^0 \mid A' \text{ convex, circled, and compact in } \sigma(E', E)\}.$$

A special intermediate topology will be denoted k; it has for a basis the collection

$$\{(A')^0 \mid A' \text{ convex, circled, and compact in } \tau(E', E)\}.$$

The topology k and others, similarly generated, are discussed in [1].

Let E, E', F, and F' be vector spaces over the same scalar field; let E and E' be in duality, likewise F and F'. Let u and v be compatible topologies for E and F, respectively. We denote by $L(E_u, F_v)$ the vector space of all continuous linear transformations of E_u into F_v. It will sometimes be convenient to say, of some $T \in L(E_u, F_v)$, that T is (u, v)-continuous. Similarly, of a subset $\mathscr{T} \subset L(E_u, F_v)$, the sentence '$\mathscr{T}$ is (u, v)-equicontinuous' will have the obvious meaning. It is known that $L(E_u, F_v) \subset L(E_\sigma, F_\sigma)$ [4; Prop. 6 and Cor., p. 103], but that the reverse inclusion may not hold. L(E, F) will hereafter denote the space $L(E_\sigma, F_\sigma)$. A necessary and sufficient condition that $T \in L(E, F)$ is that there exists a unique element $T' \in L(F', E')$, called the *adjoint* of T, with the property that $(y', Tx) = (T'y', x)$ for all $x \in E$, $y' \in F'$ [4; Prop. 1 and Cor., pp. 100, 101]. $(T')'$ is then the same as T. The following well-known statement is equivalent to the definition of continuity (NASC means 'necessary and sufficient condition'):

LEMMA 1. *Let* $T \in L(E, F)$. *A NASC that* $T \in L(E_u, F_v)$ *is that for each* $V \in v$, *there exists some* $U \in u$ *such that* $T'(V^0) \subset U^0$.

(Here $T'(V^0) = \{T'y' \mid y' \in V^0\}$. Similar algebraic notations will be used throughout.)

THEOREM 1. $L(E, F) = L(E_\tau, F_\tau) = L(E_k, F_\sigma) = L(E_k, F_k)$.

Proof. It has already been noted that L(E, F) contains the other spaces. $L(E, F) = L(E_\tau, F_\tau)$ by [4; Prop. 7, p. 104]. That $L(E, F) \subset L(E_k, F_\sigma)$ is evident from the fact that k is a stronger topology than σ for the space E. Now let $T \in L(E, F)$; we must show that $T \in L(E_k, F_k)$. But $T' \in L(F', E') = L(F'_\tau, E'_\tau)$; therefore, if $K' \subset F'_\tau$ is convex, circled, and compact, so is $T'(K')$ as a subset of E'_τ, by the linearity and (τ, τ)-continuity of T'. Thus $(T'(K'))^0$ is a neighborhood in E_k, and the criterion of Lemma 1 is satisfied. □

LEMMA 2. *Let* $\mathscr{T} \subset L(E_u, F_v)$. *A NASC that* $\mathscr{T}$ *be* (u, v)-*equicontinuous is this: For each* $V \in v$, *there exists a* $U \in u$ *such that* $\mathscr{T}'(V^0) \subset U^0$.

(Here $\mathscr{T}'$ denotes the set of adjoints to the elements of $\mathscr{T}$, and

$$\mathscr{T}'(V^0) = \{T'y' \mid T' \in \mathscr{T}', \ y' \in V^0\}.)$$

This lemma is analogous to Lemma 1; it is stated in [4; Ex. 8, p. 107]. □

A subset $A \subset E$ is *bounded* if (x', A) is a bounded set of scalars for each $x' \in E'$. A mapping $T: E \to F$ is *bounded* if T(A) is bounded in F for each bounded $A \subset E$. The (linear) space of all bounded linear transformations will be denoted B(E, F). We denote by $\mathscr{L}(E, F)$ the space of all linear mappings of E into F, and by $\mathscr{F}(E, F)$ the space of all mappings of E into F. Then

$$L(E_u, F_v) \subset L(E, F) \subset B(E, F) \subset \mathscr{L}(E, F) \subset \mathscr{F}(E, F).$$

A subset $\mathscr{T} \subset \mathscr{F}(E, F)$ will be called *pointwise bounded* if for each $x \in E$, $\mathscr{T}x$ is bounded in F, and *uniformly bounded* if for each bounded set $A \subset E$, $\mathscr{T}(A)$ is bounded in F. In general, if $\mathscr{C}$ is any family of subsets of E, $\mathscr{T}$ will be called *uniformly bounded on members of* $\mathscr{C}$ if $\mathscr{T}(C)$ is bounded in F for each $C \in \mathscr{C}$. The following facts will be useful: If a set $\mathscr{T} \subset L(E_u, F_v)$ is (u, v)-equicontinuous, then it is uniformly bounded [4; Prop. 6, p. 26]. If $\mathscr{T} \subset L(E, F)$, then $\mathscr{T}$ is pointwise bounded if and only if $\mathscr{T}'$ is pointwise bounded.

3. EQUICONTINUITY OF SUBSETS OF L(E, F)

If u is a compatible topology for E, we denote by u^0 the collection $\{U^0 | U \in u\}$. The properties of the 'antifilter base' u^0 may be found in [4; Chap. 4]; in particular we note here that if $U_i{}^0 \in u^0$ $(i = 1, 2, \cdots, n)$, then $(\bigcup_{i=1}^n U\,^0)^{00} \in u^0$; also, that each $U^0 \in u^0$ is a convex, circled, and compact subset of E' (hence closed in E'_v for each compatible v).

THEOREM 2. *Let* u *be any compatible topology for* E, *and let* $\mathscr{T} \subset L(E, F)$. *A NASC that* $\mathscr{T}$ *be* (u, σ)*-equicontinuous is that for each* $y' \in F'$, $(\mathscr{T}'y')^{00} \in u^0$.

Proof. By Lemma 2, (u, σ)-equicontinuity of $\mathscr{T}$ is equivalent to the property that for each $V^0 \in \sigma^0$, $(\mathscr{T}'(V^0))^{00} \in u^0$. Since each $y' \in F'$ is in some such V^0, the necessity of the condition is obvious. For the sufficiency, let us, without loss of generality, take $V^0 = \{y_i' | i = 1, 2, \cdots, n\}^{00}$, which is the set of all linear combinations $\Sigma_{i=1}^n a_i y_i'$, where $\Sigma_{i=1}^n |a_i| \leq 1$. Since $(\mathscr{T}'y_i')^{00} \in u^0$ for each i, we see that $\bigcup_{i=1}^n (\mathscr{T}'y_i')^{00})^{00} \in u^0$ as well. Now, if $T' \in \mathscr{T}'$ and $y' = \Sigma_{i=1}^n a_i y_i' \in V^0$, then

$$T'y' = \sum_{i=1}^n a_i T'y_i' \in \left(\bigcup_{i=1}^n \mathscr{T}'y_i'\right)^{00} = \left(\bigcup_{i=1}^n (\mathscr{T}'y')^{00}\right)^{00};$$

the last member is therefore $(\mathscr{T}'(V^0))^{00}$, and is a member of u^0. □

A *finite-dimensional subset* of a linear space is any set contained in a finite-dimensional subsapce.

COROLLARY 2A. *A NASC that* $\mathscr{T} \subset L(E, F)$ *be* (σ, σ)*-equicontinuous is that, for each* $y' \in F'$, $\mathscr{T}'y'$ *is a bounded, finite-dimensional subset of* E'.

Proof. Here the topology u of Theorem 2 is $\sigma(F, F')$, and, since all members of σ^0 are bounded and finite-dimensional, the necessity of the condition is clear. For the sufficiency, we must verify that $(\mathscr{T}'y')^{00} \in \sigma^0$. Let $y_1', y_2', \cdots, y_n'$ span the linear subspace $F_n' \subset F'$, where F_n' contains $\mathscr{T}'y'$, and let H' be the convex, circled hull of $\{y_1', y_2', \cdots, y_n'\}$. Since $\mathscr{T}'y'$ is bounded and lies in F_n', there exists an $\alpha > 0$ such that $\alpha H' \supset \mathscr{T}'y'$. Further, $\alpha H'$ is the convex circled hull of $\{\alpha y_1', \alpha y_2', \cdots \alpha y_n'\}$. Now $\{\alpha y_1', \alpha y_2', \cdots, \alpha y_n'\}^0 \in \sigma$, hence $(\mathscr{T}'y')^0$, a larger set, is also in σ. Thus $(\mathscr{T}'y')^{00} \in \sigma^0$. □

We denote the space $\mathscr{F}(E, F)$, fitted with the topology of uniform convergence in F_v on all finite sets of E ('simple convergence'), by $\mathscr{F}_s(E, F_v)$. With the topology of uniform convergence in F_v on all convex circled compact subsets of E_u, it is denoted by $\mathscr{F}_k(E_u, F_v)$. Similar notations will be used for subspaces of $\mathscr{F}(E, F)$, for example, $B_s(E, F_v)$. We remark here that $\mathscr{L}(E, F)$ is a closed linear subspace of

$\mathscr{F}_s(E, F_v)$ and of $\mathscr{F}_k(E_u, F_v)$, that is, the simple limit of a net of linear mappings is again linear.

LEMMA 3. *Let* $\mathscr{T} \subset B(E, F)$, *and let* $\mathscr{T}$ *be uniformly bounded on all compact subsets of* $E_\mathcal{T}$. *Then the closure of* $\mathscr{T}$ *in* $\mathscr{F}_s(E, F_\mathcal{T})$ *lies in* $B(E, F)$.

Proof. We denote the closure by $\overline{\mathscr{T}}$; by the remark preceding this lemma, $\overline{\mathscr{T}} \subset \mathscr{L}(E, F)$. Suppose that $T \in \overline{\mathscr{T}}$, but that T is not bounded. Then there is a sequence $\{x_n\}$ in $E_\mathcal{T}$ which converges to θ, and some $y' \in F'$, such that $|(y', Tx_n)| > n$ for all n. Since $\{\theta, x_1, x_2, \cdots, x_n, \cdots\}$ is compact in $E_\mathcal{T}$, there exists an $M > 0$ such that $|(y', Ux_n)| < M$ for all $U \in \mathscr{T}$ and all n. Then $|(y', (T - U)x_n)| > n - M$ for all $U \in \mathscr{T}$ and all n, denying that $T \in \overline{\mathscr{T}}$. □

E_u will be said to have the *convex compactness property* if, whenever A is compact in E_u, A^{00} is also compact. Any *quasi-complete* space (that is, any space in which the closed bounded sets are complete) has this property. For example, if $E_\mathcal{T}$ is a t-space, then E'_σ has the convex compactness property, because the compact sets of E'_σ are precisely the closed bounded sets [4; p. 65].

LEMMA 4. *If* E_u *has the convex compactness property, and if* $\mathscr{T} \subset L(E, F)$ *is pointwise bounded, then* $\mathscr{T}$ *is uniformly bounded.*

Proof. Suppose, to the contrary, that for some sequence $\{T_n\} \subset \mathscr{T}$, some $y' \in F'$, and some sequence $\{x_n\} \subset E_\mathcal{T}$ which converges to θ, we have $|(y', T_n x_n)| > n$ for all n. The set $\{\theta, x_1, \cdots, x_n, \cdots\}$ is compact in $E_\mathcal{T}$, hence also in E_u, and therefore $\{\theta, x_1, \cdots, x_n, \cdots\}^{00}$ is compact in E_u by the hypothesis. Then

$$K^0 = \{\theta, x_1, \cdots, x_n, \cdots\}^0$$

is a neighborhood in $E'_\mathcal{T}$ and absorbs the bounded set $\{T'_n y'\}$; that is, there exists an $\alpha > 0$ such that $\alpha K^0 \supset \{T'_n y'\}$. But this conflicts with the assumption that $|(T'_n y', x_n)| = |(y', T_n x_n)| > n$ for all n. □

For any $\mathscr{T} \subset \mathscr{F}(E, F)$, we denote by $\mathscr{T}_c$ the convex circled hull of $\mathscr{T}$.

THEOREM 3. *Let* $\mathscr{T} \subset L(E, F)$. *A NASC that* $\mathscr{T}$ *be* (k, σ)-*equicontinuous is that* $\mathscr{T}'_c$ *have compact closure in* $\mathscr{L}_s(F', E'_\mathcal{T})$.

Proof. Sufficiency: For each $y' \in F'$, the mapping $y': \mathscr{L}_s(F', E'_\mathcal{T}) \to E'$ $(y': T' \to T'y')$ is continuous and linear, hence $\overline{\mathscr{T}'_c} y'$ is convex, circled, and compact in $E'_\mathcal{T}$. Then $(\mathscr{T}'_c y')^{00}$ is contained in $\overline{\mathscr{T}'_c} y'$, and is convex, circled, and compact in E', fulfilling the condition of Theorem 2. *Necessity:* If $\mathscr{T}$ is (k, σ)-equicontinuous, then the same is true of $\mathscr{T}_c$; therefore we can assume that $\mathscr{T}$ is convex and circled. From Theorem 2 we know that for each $y' \in F'$, $\overline{\mathscr{T}'y'}$ (closure in $E'_\mathcal{T}$) is compact in $E'_\mathcal{T}$. Therefore $\mathscr{T}'$ may be embedded in the product space $\Pi_{y' \in F'} (\overline{\mathscr{T}'y'})$, which, by Tychonoff's theorem, is a compact subset of $\mathscr{F}_s(F', E'_\mathcal{T})$, and hence $\overline{\mathscr{T}'}$ is compact in $\mathscr{F}_s(F', E'_\mathcal{T})$. But $\overline{\mathscr{T}'} \subset \mathscr{L}(F', E')$, by the remark preceding Lemma 3. □

COROLLARY 3A. *Let* $\mathscr{T} \subset L(E, F)$, *and let one of the conditions* (a) *to* (d) *below hold. Then a NASC that* $\mathscr{T}$ *be* (k, σ)-*equicontinuous is that* $\mathscr{T}'_c$ *have compact closure in* $B_s(F', E'_\mathcal{T})$.

(a) $\mathscr{T}'$ *is uniformly bounded on the compact subsets of* $F'_\mathcal{T}$;

(b) $F_\mathcal{T}$ *and* $E_\mathcal{T}$ *are both* t-*spaces;*

(c) $F'_\mathcal{T}$ *is semicomplete* [7; p. 497];

(d) *for some compatible* u, F'_u *has the convex compactness property.*

Proof. The sufficiency statement is but a weakening of Theorem 3. For the necessity, we know already that $\mathscr{T}'_c$ has compact closure in $\mathscr{L}_s(F', E')$; we need only show that $\overline{\mathscr{T}'_c} \subset B(F', E')$. This follows from Lemma 3, once it is proved that each of the conditions (a) to (d), together with the (known) pointwise boundedness of $\mathscr{T}'_c$, implies the uniform boundedness of $\mathscr{T}'_c$ on compact sets of F'_τ. The conditions (a) to (d) are merely a representative list of circumstances, by no means independent of each other, which assure at least this. (a) does so directly. In case of (b), $\mathscr{T}_c$, being pointwise bounded, is (τ, τ)-equicontinuous [4; Theorem 2, p. 27]. Hence, by Lemma 2, $\mathscr{T}'_c(V^0) \in \tau^0$ for each $V^0 \in \tau^0$; this is precisely uniform boundedness, since the antifilters in the duals of t-spaces are made up of the bounded convex circled closed sets, and uniform boundedness of $\mathscr{T}'_c$ is even stronger than (a). (c) is a sufficient condition for the implication: pointwise boundedness of $\mathscr{T}'_c$ implies uniform boundedness of $\mathscr{T}'_c$ [6; p. 498]. (d) suffices for the same implication, by virtue of Lemma 4. ☐

COROLLARY 3B. *If* F'_τ *is a* t-*space, a NASC that* $\mathscr{T} \subset L(E, F)$ *be* (k, σ)-*equicontinuous is that* $\mathscr{T}'_c$ *have compact closure in* $L_s(F', E'_\tau)$.

Proof. In this case, $\mathscr{T}'_c$, being pointwise bounded, is equicontinuous in $L(F'_\tau, E'_\tau)$, hence its closure also lies in $L(F', E')$ and is compact in $L_s(F', E'_\tau)$ [4; Corollary, p. 23]. ☐

Even if F'_τ is not a t-space, the conclusion of Corollary 3B holds if $\mathscr{T}'$, and hence $\mathscr{T}'_c$, is equicontinuous in $L(F'_\tau, E'_\tau)$, that is, (τ, τ)-equicontinuous.

THEOREM 4A. *Let* $\mathscr{T} \subset L(E, F)$, *let* E'_τ *have the convex compactness property, and let* $\mathscr{T}'$ *be equicontinuous in* $L(F'_\tau, E'_\tau)$. *Let* $\overline{\mathscr{T}'}$ *denote the closure of* $\mathscr{T}'$ *in* $L_s(F', E'_\tau)$. *Then the following statements are equivalent:*

(a) $\mathscr{T}$ *is* (k, k)-*equicontinuous;*

(b) $\mathscr{T}$ *is* (k, σ)-*equicontinuous;*

(c) $\overline{\mathscr{T}'}$ *is compact in* $L_s(F', E'_\tau)$.

Proof. (a) implies (b) by comparison of topologies, and (b) implies (c) by Corollary 3B and the remark following Corollary 3B. To obtain (a) from (c), we first note that since $\mathscr{T}'$ is (τ, τ)-equicontinuous, so is its closure $\overline{\mathscr{T}'}$ [4; Prop. 4, p. 23]. For equicontinuous sets in $L(F'_\tau, E'_\tau)$, the uniform structure induced by $L(F'_\tau, E'_\tau)$ in its compact-open topology is identical with that induced by $L_s(F', E'_\tau)$ [2; Prop. 15, p. 35]; hence $\overline{\mathscr{T}'}$ is compact in the compact-open topology. Then $\overline{\mathscr{T}'}(C')$ has compact closure in E'_τ for each convex circled compact subset $C' \subset F'$ [2; Corollary, p. 44], and, by the convex compactness property of E'_τ, $(\overline{\mathscr{T}'}(C'))^{00}$ is also compact. Thus the subset of $L(E, F)$ whose adjoints form $\overline{\mathscr{T}'}$, and *a fortiori* $\mathscr{T}$ itself, are (k, k)-equicontinuous, by the criterion of Lemma 2. ☐

THEOREM 4B. *If, in addition to the hypotheses and notation of Theorem* 4A, *we ask that* F'_τ *have the convex compactness property, and denote by* $\overline{\mathscr{T}'}^k$ *the closure of* $\mathscr{T}'$ *in* $L_k(F'_\tau, E'_\tau)$, *then the following three statements are equivalent to* (a), (b), (c) *of Theorem* 4A:

(d) $\overline{\mathscr{T}'}^k$ *is compact in* $L_k(F'_\tau, E'_\tau)$;

(e) $\overline{\mathscr{T}'}^k$ *is compact in* $L_s(F', E'_\tau)$;

(f) $\overline{\mathscr{T}'}$ *is compact in* $L_k(F'_\tau, E'_\tau)$.

Proof. The reasoning here is direct, and partially repeats the proof of Theorem 4A; it is necessary to notice that the 'compact-open topology' mentioned there is the topology of L_k. We omit the details. ☐

4. EQUICONTINUITY OF SUBSETS OF L(E, E)

In the case where $E = F$, we may speak of the *algebra* $L(E, E)$, and it is possible to speak of (u, u)-continuity and (u, u)-equicontinuity, for any compatible topology u on E. In Theorem 4A, a situation is described where, if $\mathscr{T}$ is (u, σ)-equicontinuous, it is also (u, u)-equicontinuous (u being k, in that case). Here we shall give, for the case $E = F$, an algebraic characterization of this situation (Theorem 5).

Following Dixmier [8; p. 388], we denote by $\{x', x\}$ the linear transformation of rank 1 defined by the rule $\{x', x\}: y \to (x', y)x$ (here, of course, $x' \in E'$, $x \in E$). It may be calculated directly that the adjoint to $\{x', x\}$ is $\{x', x\}': y' \to (y', x)x'$.

LEMMA 5. *If* u *is a compatible topology for* E, *if* $y' \in E'$, *and if* $U^0 \in u^0$, *there exists a* (u, u)-*equicontinuous subset* $\mathscr{T} \subset L(E, E)$ *such that* $\mathscr{T}'y' = U^0$.

Proof. Choose some $x_0 \in E$ such that $(y', x_0) = 1$, and form

$$\mathscr{T} = \{\{x', x_0\} \mid x' \in U^0\}.$$

Then

$$\mathscr{T}'y' = \{\{x', x_0\}'y' \mid x' \in U^0\} = \{(y', x_0)x' \mid x' \in U^0\} = U^0.$$

To show that $\mathscr{T}$ is (u, u)-equicontinuous, we apply Lemma 2 and let $V^0 \in u^0$. Then $\mathscr{T}'(V^0) = \{(y', x_0)x' \mid y' \in V^0, x \in U^0\}$, which, since V^0 is bounded, is contained in some homothetic image of U^0, say αU^0, and $\alpha U^0 \in u^0$. □

Let us denote the class of (u, v)-equicontinuous subsets of $L(E_u, F_v)$ by $\mathcal{E}(u, v)$. It follows from Lemma 5 that the topology on E_u can be reconstructed from knowledge of $\mathcal{E}(u, u)$; indeed, a basis (at θ) for u is given by $\{(\mathscr{T}'y')^0 \mid \mathscr{T} \in \mathcal{E}(u, u), y' \in F'\}$. Furthermore, if v is a stronger compatible topology than u, there exists an element $\mathscr{T}$ in $\mathcal{E}(v, v)$ which is not in $\mathcal{E}(u, u)$, namely $\{\{x', x_0\} \mid x' \in V^0\}$, where x_0 is any nonzero element of E, and $V^0 \in v^0$, $V^0 \notin u^0$. The inclusion $\mathcal{E}(u, u) \subset \mathcal{E}(v, v)$ does not follow, however. It may happen that there exists a single transformation $T \in L(E_u, E_u)$ which is not a member of $L(E_v, E_v)$. Then $\{T\} \in \mathcal{E}(u, u)$, but $\{T\} \notin \mathcal{E}(v, v)$. Even if $L(E_u, E_u) = L(E_v, E_v)$, which is the case when u and v are drawn from $\{\sigma, k, \tau\}$, the above inclusion is only a conjecture. Lemma 5 implies that if the inclusion is true, then it is proper.

THEOREM 5. *The following statements are equivalent:*

(1) $\mathcal{E}(u, \sigma) = \mathcal{E}(u, u)$;

(2) $\mathscr{T} \in \mathcal{E}(u, \sigma)$, $\mathscr{S} \in \mathcal{E}(u, \sigma) \Rightarrow \mathscr{S}\mathscr{T} \in \mathcal{E}(u, \sigma)$;

(3) $\mathscr{T} \in \mathcal{E}(u, \sigma)$, $\mathscr{S} \in \mathcal{E}(u, u) \Rightarrow \mathscr{S}\mathscr{T} \in \mathcal{E}(u, \sigma)$.

Proof. (1) $\Rightarrow$ (2) is immediate from the definition of (u, u)-equicontinuity, and (2) $\Rightarrow$ (3) follows from $\mathcal{E}(u, u) \subset \mathcal{E}(u, \sigma)$ (since σ is weaker than u). Now we assume (3), and let $\mathscr{T} \in \mathcal{E}(u, \sigma)$. To prove $\mathscr{T} \in \mathcal{E}(u, u)$, thereby proving (1), we let $V^0 \in u^0$, and proceed to show that $\mathscr{T}'(V^0)$ is contained in a member of u^0. By Lemma 5, for each $y' \in E'$, we can find an $\mathscr{S} \in \mathcal{E}(u, u)$ such that $\mathscr{S}'y' = V^0$. Then $\mathscr{S}\mathscr{T} \in \mathcal{E}(u, \sigma)$, and, by Theorem 2, $(\mathscr{S}\mathscr{T})'y'$ is contained in an element of u^0. But

$$(\mathscr{S}\mathscr{T})'y' = \mathscr{T}'\mathscr{S}'y' = \mathscr{T}'V^0. \quad \square$$

5. A TOPOLOGICAL LEMMA

THEOREM 6. *Let* E *be a set of points with two Hausdorff topologies* u *and* v. *Let* $w = \sup(u, v)$ *and* $t = \inf(u, v)$. *Then, of the four statements given below,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If, in addition,* u *and* v *satisfy the first axiom of countability, the four statements are equivalent.*

(1) E_t *is a Hausdorff space;*

(2) *if* $\{x_\alpha \mid \alpha \in A\}$ *is a converging net in both* E_u *and* E_v, *then*

$$\text{u-lim}_\alpha\, x_\alpha = \text{v-lim}_\alpha\, x_\alpha\,;$$

(3) *the family of compact subsets of* E_w *is exactly*

$$\{K_1 \cap K_2 \mid K_1 \text{ compact in } E_u,\ K_2 \text{ compact in } E_v\}\,;$$

(4) K *is compact in both* E_u *and* E_v *if and only if* K *is compact in* E_w.

Proof. (The terminology here is that of [9; pp. 65ff].) (1) $\Rightarrow$ (2) follows from the uniqueness of a limit in E_t. (2) $\Rightarrow$ (3): If K is compact in E_w, it is compact in each of the weaker topologies u and v, and $K = K \cap K$ is the required representation. Now let $K = K_1 \cap K_2$, as in (3), and let $\mathscr{A}$ be a net in K. Some subnet $\mathscr{B}$ of $\mathscr{A}$ converges (in E_u) in the compact set K_1, and a further subnet $\mathscr{C}$ converges (in E_v) in the compact set K_2. Then $\mathscr{C}$ converges in both E_u and E_v to a point $x \in K$, implying that $\mathscr{C}$ converges to x in E_w. (3) $\Rightarrow$ (4): As before, compactness in E_w implies compactness in E_u and E_v. Conversely, if K is compact in both E_u and E_v, then $K = K \cap K$ is compact in E_w, by (3). (4) $\Rightarrow$ (1) (under the countability assumption of the theorem): Suppose that t is not a Hausdorff topology, that is, that there exist distinct points x and y in E whose t-neighborhoods always intersect. If $\{U_n(x)\}$ is a basis of u-neighborhoods of x, $\{U_n(y)\}$ the same for y, $\{V_n(x)\}$ a basis of v-neighborhoods of x, and $\{V_n(y)\}$ the same for y, then a t-basis at x is formed by $\{U_n(x) \cup V_n(x)\}$, and a t-basis at y by $\{U_n(y) \cup V_n(y)\}$. For each n, there exists a point $z_n \in [U_n(x) \cup V_n(x)] \cap [U_n(y) \cup V_n(y)]$; equivalently,

$$z_n \in [U_n(x) \cap U_n(y)] \cup [V_n(x) \cap V_n(y)] \cup [U_n(x) \cap V_n(y)] \cup [V_n(x) \cap U_n(y)]\,.$$

For sufficiently high n, the first two sets in the above union are empty, because u and v are Hausdorff topologies. We may assume, without loss of generality, that, for all n, $z_n \in U_n(x) \cap V_n(y)$. Then $\text{u-lim}_n z_n = x$, and $\text{v-lim}_n z_n = y$, and the set $K = \{x, y, z_1, z_2, \cdots, z_n, \cdots\}$ is compact in both E_u and E_v. By (4), K is compact in E_w. In K, then, $\{z_n\}$ has a w-converging subsequence $\{z_{n_k}\}$, with unique limit z. Since $x = \text{u-lim}_k z_{n_k}$, and w is a stronger topology than u, z = x. Similarly, z = y, which is impossible. □

6. COMPACTNESS IN A RING OF OPERATORS

The adjoint mapping $T \to T'$ of L(E, F) onto L(F', E') is an isomorphism of the vector spaces; hence any locally convex topology on one of them induces 'by transportation' a locally convex topology on the other. If the space $L_s(E, F_\mathcal{T})$ is denoted by L_s, we shall denote by $L_{s'}$ the space L(E, F) supplied with the topology s' obtained by transportation from $L_s(F', E'_\mathcal{T})$. In other words, $\lim_\alpha T_\alpha = T$ in $L_{s'}$

means that $\lim_\alpha T_\alpha' = T'$ in $L_s(F', E_\tau')$. The conditions on $\mathscr{T}'$ in Theorems 3 and 4 may now be interpreted as topological conditions on $\mathscr{T}$ as a subset of $L_{s'}$.

Let L' denote the following space of linear functions on $L(E, F)$: for each $y' \in F'$ and each $x \in E$, denote by $[y', x]$ the functional $[y', x]$: $T \to (y', Tx)$; then L' is the set of all finite linear combinations of such functionals. It is proved in [4; Prop. 11, p. 77] that the topology s of L_s is compatible with the duality of L and L'. By the same token, linear combinations of functionals of the form $[x, y']$: $T' \to (T'y', x)$ make up the dual space of $L_s(F', E_\tau')$, so that, by transportation, s' is also compatible with the duality of L and L'. Finally, if $s^+ = \sup(s, s')$, it is clear that s^+ is also compatible with the duality (because of Mackey's theorem that there exist upper and lower bounds for the lattice of compatible topologies). It should be noted that when E and F are infinite-dimensional spaces, these topologies are distinct, since s and s' are incomparable (Dixmier's proof [8; p. 406], given for E = F, a Hilbert space, is valid without essential change).

THEOREM 7. *Let* $K \subset L(E, F)$. *Then* K *is compact in* L_{s^+} *if and only if* K *is compact in both* L_s *and* $L_{s'}$.

Proof. By Theorem 6, it suffices to show that $\inf(s, s')$ is a Hausdorff topology for L. But since s and s' are each compatible with the same duality of L and L', $\inf(s, s')$ is at least as strong as $\sigma(L, L')$, the Mackey lower bound, which is a Hausdorff topology. □

Now let E be a Hilbert space, and let F = E. Then E_τ (= E_τ') is the usual normed space, is complete, and therefore has the convex compactness property. Also ('Banach-Steinhaus Theorem'), any pointwise bounded subset of L(E, E) is (τ, τ)-equicontinuous.

THEOREM 8. *Let* $\mathscr{T} \subset L(E, E)$ (E *a Hilbert space*).

(1) $\mathscr{T}$ *has compact closure in* L_s *if and only if* $\mathscr{T}'$ *is* (k, σ)- *or* (k, k)-*equicontinuous in* L(E, E).

(2) $\mathscr{T}$ *has compact closure in* $L_{s'}$ *if and only if* $\mathscr{T}$ *is* (k, σ)- *or* (k, k)-*equicontinuous in* L(E, E).

(3) $\mathscr{T}$ *has compact closure in* L_{s^+} *if and only if both* $\mathscr{T}$ *and* $\mathscr{T}'$ *are* (k, σ)- *or* (k, k)-*equicontinuous in* L(E, E).

(4) *If, in any one of the topologies* s, s', *and* s^+, *two subsets* $\mathscr{T}$ *and* $\mathscr{S}$ (of L(E, E)) *have compact closure, then* $\mathscr{S}\mathscr{T}$ *has compact closure.*

(5) L_s, $L_{s'}$, L_{s^+} *all have the convex compactness property.*

Proof. (1) and (2) are restatements of Theorem 4 for this case. If both $\mathscr{T}$ and $\mathscr{T}'$ are (k, σ)-equicontinuous, then by (1) and (2), $\overline{\mathscr{T}}^{s'}$ is s'-compact, and $\overline{\mathscr{T}}^{s}$ is s-compact (bars mean closures in the indicated topologies). Since $\inf(s, s')$ is a Hausdorff topology, we may apply (1) $\Rightarrow$ (3) of Theorem 6 to conclude that $\overline{\mathscr{T}}^{s} \cap \overline{\mathscr{T}}^{s'}$ is s^+-compact. But $\overline{\mathscr{T}}^{s^+}$ is contained in $\overline{\mathscr{T}}^{s} \cap \overline{\mathscr{T}}^{s'}$, and it is s^+-closed, hence s^+-compact, which proves the nontrivial part of (3). (4) is a consequence of Theorem 5, since (k, σ)- and (k, k)-equicontinuity are equivalent here. (5) is true because the convex circled extension of an equicontinuous set of linear mappings is again equicontinuous. □

All of Theorem 8 is also valid, respectively, for the corresponding three 'ultrafort' topologies discussed by Dixmier [8; p. 406], in view of his proof that they are, respectively, equivalent to s, s', and s^+ on pointwise bounded sets of L(E, E). Also, it can be shown by an example that Theorem 8 is not vacuous; for example, in the notation of Section 4, let $\{\{x', x_0\} \mid \|x'\| \leq 1\}$ be the set $\mathscr{T}$, where the functionals x' are of course again elements of the Hilbert space E. Then $\mathscr{T}x$ is a bounded 1-dimensional set in E, for each $x \in E$; hence $\mathscr{T}'$ is (k, σ)-equicontinuous (because it is (σ, σ)-equicontinuous, by Corollary 2A), and $\mathscr{T}$ has compact closure in L_s. But it can be verified by the criterion of Lemma 2 that $\mathscr{T}$ itself is not (k, σ)-equicontinuous, whence $\mathscr{T}$ does not have compact closure in $L_{s'}$ or in L_{s^+}.

REFERENCES

1. R. Arens, *Duality in linear spaces*, Duke Math. J. 14 (1947), 787-794.

2. N. Bourbaki, *Eléments de mathématique*. X. Part 1, Book III. *Topologie générale*. Chapter X. *Espaces fonctionnels; dictionnaire*, Actualités Sci. Ind., no. 1084, Paris (1949).

3. ———, *Sur certains espaces vectoriels topologiques*, Ann. Inst. Fourier Grenoble 2 (1950), 5-16 (1951).

4. ———, *Eléments de mathématique*. XVIII. Part 1, Book V. *Espaces vectoriels topologiques*, Chapters III to V, Actualités Sci. Ind. no. 1229, Paris (1955).

5. J. Dieudonné, *Natural homomorphisms in Banach spaces*, Proc. Amer. Math. Soc. 1 (1950), 54-59.

6. ———, *Recent developments in the theory of locally convex vector spaces*, Bull. Amer. Math. Soc. 59 (1953), 495-512.

7. J. Dieudonné and L. Schwartz, *La dualité dans les espaces $\mathscr{F}$ et $\mathscr{L}\mathscr{F}$*, Ann. Inst. Fourier Grenoble 1 (1949), 61-101 (1950).

8. J. Dixmier, *Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert*, Ann. of Math. (2) 51 (1950), 387-408.

9. J. L. Kelley, *General topology*, New York, D. Van Nostrand Co. (1955).

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