ON THE SHEETED STRUCTURE OF COMPACT LOCALLY AFFINE SPACES

Louis Auslander

INTRODUCTION

Let M^n be an n-dimensional, compact, locally affine space; that is, let M^n carry a complete affine connection with curvature and torsion tensors equal to zero. It is well known (see [1]), that any locally affine space can be realized in the following manner. Let Γ be the fundamental group of M^n. Then the affine connection on M^n determines an imbedding of Γ in the group $A(n)$ of affine transformations of the n-dimensional affine space A^n. Further, the orbit space A^n/Γ is homeomorphic to M^n. Let T denote the subgroup of Γ consisting of all pure translations. Then T is a free abelian group on a finite number of generators. Let $h(\Gamma) = \Gamma/T$. Then $h(\Gamma)$ is called the holonomy group of Γ. The purpose of this paper is to prove the following three theorems:

THEOREM 1. *Let* T *be a free abelian group on* s *generators* ($s \geq 1$). *Assume also that* $h(\Gamma)$ *contains no elements of finite order. Then* M^n *is a fiber bundle over a compact locally affine space* X *with the* s*-dimensional torus as fiber. Further, the fundamental group of* X *is isomorphic to* $h(\Gamma)$.

THEOREM 2. *Let* T *be a free abelian group on* s *generators* ($s \geq 1$). *Then there exists a mapping* $p: M \to X$, *where* X *is a compact space (not necessarily a manifold) with the following properties:*

I. *For all* $x \in X$, $p^{-1}(x)$ *is a compact,* s*-dimensional manifold which can be given a Riemann metric with zero curvature and torsion.*

II. *The mapping* p *satisfies the hypothesis required for applying the Fáry spectral sequence* (see [2]).

In [3], Zassenhaus defined the radical R of a discrete matrix group Γ as the maximal solvable normal subgroup, and he proved that R is unique.

THEOREM 3. *Let* Γ *be the fundamental group of a compact locally affine space* M, *and assume that* Γ *has a nontrivial radical. Then there exists a mapping* $p: M \to X$, *where* X *is a compact space (not necessarily a manifold) and the preimage of each point of* X *under* p *is a compact manifold with a torus as covering space.*

The paper concludes with an example of a locally affine manifold which satisfies the hypothesis of Theorem 2, but not the hypothesis of Theorem 1.

Received October 15, 1957.

This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 49(638)-62. Reproduction in whole or in part is permitted for any purpose of the United States Government.

1. GENERAL CONSIDERATIONS

Let $t_1, \cdots, t_s$ be a basis for T. Then through each point of A^n the basis $t_1, \cdots, t_s$ determines a unique s-dimensional plane E^s, and all the planes thus determined are parallel. Let $\mathfrak{E}$ denote this family of parallel planes. Then $\mathfrak{E}$ determines a projection $p_*: A^n \to A^{n-s}$; and Γ may be considered as acting on A^{n-s}. For, if $\gamma \in \Gamma$, then $\gamma t_i \gamma^{-1} = \Sigma a_{ij} t_j$ (i, j = 1, ⋯, s). Hence for each E^s in $\mathfrak{E}$, Γ maps E^s either onto itself or onto another element of $\mathfrak{E}$. We denote the action of Γ on A^{n-s} by $P_*(\Gamma)$. Then $P_*(\Gamma) \subset A(n - s)$. We choose a coordinate system in A^n in such a way that the first s coordinates span E^s, and we represent the points of A^n by column vectors. In terms of homogeneous coordinates, every element of Γ has a matrix representation of the form

$$\begin{pmatrix} A & X & t_1 \\ 0 & B & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this array, A is an s-by-s nonsingular matrix, B an (n - s)-by-(n - s) nonsingular matrix; X is any s-by-(n - s) matrix, t_1 is a 1-by-s column vector, t_2 is a 1-by-(n - s) column vector, and the last row of the matrix has all zero entries except in the last column, where there is a 1. Then the mapping P_* may be explicitly represented by

$$P_* \begin{pmatrix} A & X & t_1 \\ 0 & B & t_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} B & t_2 \\ 0 & 1 \end{pmatrix}.$$

LEMMA 1. *If* $h(\Gamma)$ *has no elements of finite order, then* $P_*(\Gamma)$ *is isomorphic to* $h(\Gamma)$; *equivalently: an element of* Γ *acts trivially on* A^{n-s} *if and only if it is in* T.

Proof. By a straight-forward calculation one can show that P_* is a homomorphism. Let γ in Γ be in the kernel of P_*. We shall show that γ is a pure translation. Consider γ^n (n = 1, 2, ⋯). If t_n represents the translation components of γ^n, then $t_n = \Sigma_{i=1}^{s} a_{ni} t_i$ for all n. Hence, by multiplying γ^n on the left by a properly chosen element of T, we obtain an infinite sequence of elements of Γ with bounded translational components. Hence, since Γ can have no accumulation point or fixed points, γ^n is in T, from some n on. But since $h(\Gamma)$ has no elements of finite order and $h(\Gamma)$ is isomorphic to the group of n-by-n matrices $\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$, it follows that γ is a pure translation. But any pure translation in Γ is in the kernel of P_*. Therefore $P_*(\Gamma)$ is isomorphic to $h(\Gamma)$.

LEMMA 2. *Under the hypothesis of Lemma 1,* $A^{n-s}/P_*(\Gamma)$ *is a compact Hausdorff manifold.*

Proof. Assume that there exists an x in A^{n-s} and an h in $P_*(\Gamma)$ such that $h(x) = x$. Let $\gamma \in P_*^{-1}(h)$. Then for all x' in A^n such that $p_* x' = x$, $p_*(\gamma^n x') = p_* x$. Hence, reasoning as in the proof of Lemma 1, we see that γ must be a pure translation and that h is the identity element of $P_*(\Gamma)$. Therefore $P_*(\Gamma)$ operates without fixed points on A^{n-s}.

Assume that $h_1, \cdots, h_n, \cdots \in P_*(\Gamma)$, and that an x in A^{n-s} exists such that $\{h_i(x)\}$ $(i = 1, 2, \cdots)$ is a Cauchy sequence. Let x' in A^n be such that $p_*x' = x$. Then again we can find γ_i in Γ such that $P_*\gamma_i = h_i$ for all i, and such that the $\gamma_i x'$ are in a bounded domain of A^n. This contradicts the hypothesis on Γ.

In a similar way, we can show that $A^{n-s}/P_*(\Gamma)$ is a compact Hausdorff space.

Definition. For any m_1 and m_2 in A^n/Γ we say that m_1 is equivalent to m_2 $(m_1 \sim m_2)$ if there exist pre-images $\tilde{x}_1$ and $\tilde{x}_2$ in A^n of m_1 and m_2, respectively, with the property that $p_*\tilde{x}_1 = p_*\tilde{x}_2$. We denote by X the identification space for A^n/Γ under $\sim$, by p the projection of A^n/Γ onto X, and by Γ_0 the kernel of P_* acting on Γ.

LEMMA 3. $\Gamma_0 \supset T$, *and* $X = A^{n-s}/(\Gamma/\Gamma_0)$. *If* $P_*(\Gamma)$ *has no elements of finite order, then* $\Gamma_0 = T$ *and* $X = A^{n-s}/P_*(\Gamma)$.

The proof of this lemma is straight-forward, and it will be omitted. Together, the three lemmas supply a proof of Theorem 1 of the Introduction.

2. PROOF OF THEOREM 2

LEMMA 4. *Let* p *denote the projection of* A^n/Γ *onto* X. *Then for* x *in* X, $p^{-1}(x)$ *is a compact locally euclidean manifold (Riemann manifold with zero curvature and torsion).*

Proof. We now have the commutative diagram

$$\begin{array}{ccc} A^n & \xrightarrow{p_*} & A^{n-s}, \\ p_1 \downarrow & & \downarrow p_2 \\ A^n/\Gamma & \xrightarrow{p} & X \end{array}$$

where p_1 and p_2 are defined in the obvious manner. For $x \in X$, let a in A^{n-s} be such that $p_2(a) = x$. Let $p_*^{-1}(a) = E_a^s$, and let Γ_a be the subgroup of Γ which maps E_a^s onto itself. Then $p_1(p_*^{-1}(p_2^{-1}x))$ is homeomorphic to E_a^s/Γ_a, and equals $p^{-1}(x)$. Since Γ_a contains s linearly independent translations T, Γ_a/T must be a finite group. Hence E_a^s/Γ_a can be considered as a compact, locally affine space with finite holonomy group. Hence the affine connection can be induced by a Riemann metric with zero curvature and torsion, and E_a^s/Γ_a is a compact, locally euclidean manifold.

LEMMA 5. *For each* x_0 *in* X, *there exists a neighborhood* $U(x_0)$ *so small that* $\Gamma_x \subset \Gamma_{x_0}$ *for each* x *in* $U(x_0)$. (The usefulness of such a lemma was pointed out to me by J. Milnor.)

Proof. In the proof of this lemma, we shall actually need the fact that the manifold A^n/Γ is a Hausdorff space. In terms of the action of Γ on A^n, this may be stated as follows: given nonequivalent x and y in A^n, there exist open sets $U(x)$ and $V(y)$ such that $\gamma_1 U(x) \cap \gamma_2 V(y)$ is empty for all γ_1 and γ_2 in Γ. Let us assume that the lemma is false. Then there exist E_i^s in $\mathfrak{E}$ $(i = 1, 2, \cdots)$ and an E_0^s in $\mathfrak{E}$ such that $p_*E_i^s$ converge to $p_*E_0^s$ in A^{n-s}, and with the further property that if Γ_i $(i = 1, 2, \cdots)$ leaves E_i^s fixed, there exists a γ_i in Γ_i which does not leave E_0^s fixed. Choose y_i in A^n such that $y_i \in E_i^s$ for all i, and such that the sequence of y_i converges to y_0 in E_0^s. Then new γ_i can be so chosen (since $T \subset \Gamma_i$, for all i) such that the $\gamma_i y_i$ are bounded, and hence a properly chosen subsequence of the $\gamma_i y_i$ may be assumed to converge to some $\bar{y}$. Now chose any open sets $U(\bar{y})$ and $V(y_0)$. Then there exist y_i in V and γ_i in Γ such that $\gamma_i y_i$ are in U. This contradicts the Hausdorff axiom, and hence proves the lemma unless $\bar{y} = \gamma_0 y_0$, where γ_0 is not the identity of Γ. But y_i converges to y_0, and $\gamma_0^{-1}\gamma_i y_i$ converges to y_0. This implies that the orbits of y_0 under Γ do not have the property that there exists an open set of y_0 all of whose translations under Γ are disjoint, unless $\gamma_0^{-1}\gamma_i = e$ for all i greater than some fixed N. But $\gamma_0^{-1}\gamma_i = e$ contradicts the assumption that $\gamma_i y_0$ is not in E_0^s. This proves the lemma.

LEMMA 6. *Let the kernel of* P^* *be* Γ_0. *Then there exists a dense open set* V_1 *in* X *such that for any* x *in* U_1, $p^{-1}(x)$ *has fundamental group* Γ_0.

Proof. It is easy to verify that Γ_0 is a normal subgroup of Γ. Lemma 5 and the definition of Γ_0 imply that the set U_1 of points x such that $p^{-1}(x)$ has fundamental group Γ_0 is open in X. We must now show that each point x not in U_1 has the property that every open neighborhood of x meets U_1. Let Γ_x be the subgroup of Γ leaving E_a^s fixed, where $p_2(a) = x$. Then $\Gamma_x \supset \Gamma_0$. Now the homogeneous parts of Γ_x and $h(\Gamma_x)$, constitute a finite group. Since $h(\Gamma_x)$ is a finite group, it has only a finite number of subgroups. But these subgroups determine the set of points in a neighborhood of x which are left fixed by groups other than Γ_0. Each of this finite number of groups leaves fixed a linear space of dimension less than n - s. Since there are only a finite number of subgroups, every open set containing x meets U_1.

LEMMA 7. $C_1 = X - U^1$ *is an* ANR (ANR = *absolute neighborhood retract*). *Furthermore, there exists a finite number of sets* $U_1^2, \cdots, U_r^2$ *that are open in* C_1 *and dense in* C_1, *and such that over each* U_i^2 (i = 1, ⋯, r) *we have a local product bundle. Furthermore* $C_1 - \bigcup U_i^2$ *is a closed* ANR.

Proof. The first part of the lemma follows from the fact that C_1 is locally the union of a finite number of planes. The second part can be proved by applying the preceding lemmas to these planes, one at a time.

By induction, we see that we have fulfilled the hypothesis required for applying the Fáry spectral sequence as given in [2]. This completes the proof of Theorem 2.

3. PROOF OF THEOREM 3

Let Γ be the fundamental group of M, with nontrivial radical R. Then, since R is solvable, R contains a nontrivial normal abelian subgroup. Since Γ, being invariant under all automorphisms of R, operates without fixed points, it has no elements of finite order. Hence the nontrivial normal abelian subgroup must be free abelian on s generators, for $s \geq 1$. We denote it by Z^s. Now, since Z^s is invariant under all automorphisms of R, and R is a normal subgroup of Γ, Z^s is a normal subgroup of Γ. But by [3, Theorem 12, p. 308], every normal abelian subgroup of Γ has all eigenvalues equal to 1, if A^n/Γ is compact. Hence Z^s can be simultaneously diagonalized, and it lies on a unique minimal algebraic subgroup G of A(n). Furthermore, it is easy to see that G/Z^s is compact and is homeomorphic to the s-dimensional torus. Now the group G is a normal subgroup in ΓG. This follows from the fact that $\gamma G \gamma^{-1} \supset Z^s$ for γ in Γ and $\gamma G \gamma^{-1}$ is an algebraic group. But G is the unique minimal algebraic subgroup containing Z^s; therefore $\gamma G \gamma^{-1} = G$, and G is normal in ΓG. (This argument was suggested by G. D. Mostow.) We may now construct a proof of Theorem 3 by a method analogous to that used in proving Theorem 2.

4. AN EXAMPLE

Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ & 1 & 0 & \frac{1}{2} \\ & & -1 & 0 \\ & & & 1 \end{pmatrix},$$

$$T = [A, B^2] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},$$

where all omitted entries are zero and the bracket denotes the commutator.

By straightforward calculation, we have

$$BT^sB^{-1} = T^{-s}, \quad BA^sB^{-1} = A^{-s},$$

$$B^{-1}T^sB = T^{-s}, \quad B^{-1}A^sB = A^{-s}.$$

Hence the subgroup of Γ generated by A, B^2, T is a normal subgroup. Call it Γ'. Then any element of Γ can be written as $B\gamma'$ or γ', for γ' in Γ'. Since A^n/Γ' is a compact, locally affine manifold, it is easily verified that A^3/Γ is a compact, locally affine manifold whose holonomy group contains elements of finite order.

REFERENCES

1. L. Auslander and L. Markus, *Holonomy of flat affinely connected manifolds*, Ann. of Math. (2) 62 (1955), 139-151.

2. I. Fáry, *Valeurs critiques et algebres spectrales d'une application*, Ann. of Math. (2) 63 (1956), 437-490.

3. H. Zassenhaus, *Beweis eines Satzes über diskrete Gruppen*, Abh. Math. Sem. Univ. Hamburg 12 (1938), 289-312.

Indiana University