BOUNDED CONTINUOUS FUNCTIONS ON A LOCALLY COMPACT SPACE

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1. INTRODUCTION

Let X be a compact Hausdorff space, and C[X] the collection of all bounded real-valued continuous functions on X. In the two decades since the appearance of Stone's paper [10], this has been an intensively studied mathematical object. Its interest arises in part from its rich structure; under the uniform norm topology, C[X] is a Banach space (see Myers [7]); under pointwise multiplication, it is an algebra (see Hewitt [3]); under the natural partial ordering, it is a lattice (see Kaplansky [4]). In each case, the underlying compact space X has a faithful representation within the structure of C[X], from which it can be recovered, so that C[X] may be regarded as a tool for the study of X.

When X is no longer compact, the simplicity breaks down. Let C*[X] denote the collection of bounded functions in C[X]. If X is completely regular, then there is associated with it a unique compact space βX, the Stone-Čech compactification of X, and C*[X] and C*[βX] are algebraically isomorphic; X is dense in βX, and every $f \in$ C*[X] has a bounded continuous extension to βX. An analogous pattern holds for C[X] (see [3]). In studying C*[X], we are thus again studying the algebra of all (bounded) continuous functions on a compact space. However, βX can have a very complicated structure, even when X is itself relatively nice (for example, when X is the line). When X is locally compact, something can be achieved by considering the subalgebra $C_0[X]$ of functions on X which vanish at infinity; for this is isomorphic with a fixed maximal ideal in $C[X^0]$, where X^0 is the one-point compactification of X.

In this paper, I shall deal with the full algebra C*[X] for a locally compact space X, with a new topology β; this topology was introduced, in an earlier paper [2], for the special case in which X is a group; it was there called the "strict" topology, because of its resemblance to a topology used by Beurling [1]. In Section 3, the strict topology will be defined, and its properties described; in particular, we prove that it is topologically complete. (This is a considerable improvement on [2], where it was only shown that β is sequentially complete.) In Section 4, we show that the dual space of β-continuous linear functionals in C*[X] is precisely $\mathfrak{M}$[X], the space of bounded Radon measures on X. In Section 5, we obtain a Stone-Weierstrass theorem for β-closed subalgebras of C*[X]. For the sake of completeness, we have extended the preliminary treatment to the space C*[X: E] of bounded continuous functions on the locally compact space X which take values in an arbitrary complete locally convex linear space E. In Section 6, we obtain a type of Stone-Weierstrass theorem for β-closed subspaces of C*[X: E]; this result is of particular interest since the object of study is no longer an algebra of continuous functions, but only a linear space.

2. PRELIMINARY DEFINITIONS

Before introducing the strict topology, we give certain basic definitions, and state several more or less well-known results dealing with continuous functions on

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locally compact spaces. In all that follows, X is a fixed locally compact space. Let E be a real linear space, with a complete locally convex topology described by an indexed family of semi-norms $\| \|_\nu$. C[X: E] is the real linear space of all continuous mappings from X into E. C*[X: E] is the subspace of those mappings f for which f(X) is a bounded set in E. C_0[X: E] is the subspace of C*[X: E] consisting of those mappings f that vanish at infinity; explicitly, if V is any neighborhood of the origin in E, and $f \in C_0[X: E]$, then there is a compact set $K \subset X$ such that $f(x) \in V$ for every x in X, outside of K. C_{00}[X: E] is the subspace of C_0[X: E] consisting of those f which are identically 0 in E, outside of some compact set in X. When E is taken as the real field, these spaces are simply spaces of real-valued functions defined on X, and are denoted by C[X], C*[X], C_0[X], and C_{00}[X].

A set S is said to be *σ-compact* if $S = \bigcup K_n$, where each set K_n is compact. If these sets can be chosen so that K_n is always contained in the interior of K_{n+1}, then S is said to be *regularly σ-compact.* Every σ-compact set is a subset of a regularly σ-compact set. Every regularly σ-compact set is an open F_σ.

LEMMA 1. *If* $\phi \in C_0[X]$, *then* ϕ *vanishes, except on a regularly σ-compact set in* X. *Conversely, if* S *is any regularly σ-compact set in* X, *there exists a function* $\phi \in C_0[X]$ *which takes values in* [0, 1] *and which is zero outside of* S, *but strictly positive on* S.

LEMMA 2. *If* $\{x_n\}$ *is a discrete sequence of points in* X *and* $\{c_n\}$ *is a null sequence of positive real numbers, then there is a function* $\phi \in C_0[X]$ *such that* $\phi(x_n) = c_n$.

To prove the first, observe that if $\phi \in C_0[X]$, then there is a compact set K_n such that $|\phi(x)| < 1/n$ for all $x \notin K_n$, so that ϕ vanishes except in $\bigcup K_n$. For the converse, let $S = \bigcup K_n$, and choose ϕ_n with range in [0, 1] such that $\phi_n(x) = 1$ for $x \in K_n$, and $\phi_n(x) = 0$ for x off K_{n+1}. Setting $\phi(x) = \Sigma 2^{-n} \phi_n(x)$ yields the desired function.

To prove the second, choose a sequence of compact sets K_n with $x_n \in K_n$, but with no two sets overlapping. Take ϕ_n with range in [0, 1], vanishing off K_n, and with $\phi_n(x_n) = 1$. With $\phi(x) = \Sigma c_n \phi_n(x)$, we obtain the desired function.

These lemmas guarantee the existence of certain functions in C_0[X]. In particular, if X itself is σ-compact, then there exists a function on X which vanishes at infinity, but nowhere else; and this is not true if X is not σ-compact. Note: this shows at once that a σ-compact space cannot be pseudo-compact (every continuous function is bounded) unless it is indeed compact (see [3]).

A σ-compact space is certainly paracompact; an easy argument shows the following to be true [6].

LEMMA 3. *If* X *is locally compact, paracompact, and connected, then* X *is σ-compact.*

In general, then, a locally compact paracompact space is the union of a collection of disjoint connected σ-compact spaces, and the algebra C*[X] is the full direct sum of algebras $C^*[X_\alpha]$, where X_α is σ-compact. The structure of C_0[X] is more complicated. Suppose that $X = \bigcup X_\alpha$, where each X_α is open and connected. Every function $\phi \in C_0[X]$ is then a direct sum of functions ϕ_α, where $\phi_\alpha \in C_0[X_\alpha]$; more is true, however; for it is necessary that all but a countable set of the component functions ϕ_α vanish identically, and that those that are left, ϕ_{α_k}, satisfy the condition $\lim_{k\uparrow} \sup_{x \in X_{\alpha_k}} |\phi_{\alpha_k}(x)| = 0$.

Although the space $C^*[X: E]$ is not an algebra, since no product of mappings is defined, it is naturally a $C^*[X]$ module. If $f \in C^*[X: E]$ and $\phi \in C^*[X]$, then ϕf is the mapping defined by $(\phi f)(x) = \phi(x) f(x)$. Moreover, if $\phi \in C_0[X]$, then $\phi f \in C_0[X: E]$. This property serves to characterize $C^*[X: E]$ within $C[X: E]$.

LEMMA 4. *Let* $F \in C[X: E]$, *and suppose that* $\phi F \in C_0[X: E]$ *for every* $\phi \in C_0[X]$. *Then* $F \in C^*[X: E]$.

If F were not bounded on X, then, for some index ν, there would exist a sequence $\{x_n\}$ such that $\| F(x_n) \|_\nu \geq n$ for $n = 1, 2, \cdots$. Since F is continuous on X, the sequence $\{x_n\}$ is discrete. By Lemma 2, we may choose $\phi \in C_0[X]$ so that $\phi(x) F(x) = 1$ for $x = x_1, x_2, \cdots$. Then ϕF would not belong to $C_0[X: E]$.

3. THE STRICT TOPOLOGY

The method which we use to obtain the topology β on $C^*[X: E]$, or on $C^*[X]$, is a special case of a general procedure. Let V be a vector space, with a locally convex topology τ, and let $\mathfrak{A}$ be a collection of τ-continuous linear operators on V. Then, the weak $\mathfrak{A}$-topology on V, denoted by $w(\mathfrak{A})$, is defined to be the smallest topology β on V such that each transformation $T \in \mathfrak{A}$ remains continuous as a mapping of $\langle V, \beta \rangle$ into $\langle V, \tau \rangle$. An illustration: the ordinary weak topology on V defined by the dual space of $\langle V, \tau \rangle$ is the topology $w(\mathfrak{A})$, where $\mathfrak{A}$ is the set of all transformations with finite dimension range (of finite rank). Again, Raimi [8] has studied the topology which is defined in this way on Hilbert space, choosing $\mathfrak{A}$ as the algebra of compact transformations.

In the function algebra $C^*[X]$, a useful topology is the uniform topology σ, defined by the norm $\|f\| = \sup_{x \in X} |f(x)|$. Another topology, equally familiar, is the compact open topology κ. This is locally convex, and may be defined by the seminorms $\|f\|_K = \max_{x \in K} |f(x)|$, where K ranges over the compact subsets of X. Convergence (κ) means uniform convergence on each compact set in X; convergence (σ) means uniform convergence on all of X. These topologies have their analogues in the spaces $C^*[X: E]$. The topology σ is defined by the semi-norms $\|f\|_\nu = \sup_{x \in X} \| f(x) \|_\nu$, and the compact-open topology κ by the semi-norms

$$\|f\|_{K,\nu} = \max_{x \in K} \| f(x) \|_\nu \quad ;$$

here K is a general compact set in X. If X is in fact compact, then $\kappa = \sigma$.

DEFINITION. *The strict topology* β *on* $C^*[X: E]$ *is the weak topology* $w(\mathfrak{A})$ *obtained by taking* $C^*[X: E]$ *as the space* V, σ *as the topology* τ, *and* $\mathfrak{A}$ *as the family of transformations* $f \to \phi f$ *defined by functions* $\phi \in C_0[X]$.

An equivalent description: β is the locally convex topology defined on $C^*[X: E]$ by the semi-norms

$$\|f\|_{\phi,\nu} = \|\phi f\|_\nu = \sup_{x \in X} \| \phi(x) f(x) \|_\nu$$

where ν ranges over the indexed topology on E, and ϕ ranges over $C_0[X]$. Note that the compact-open topology κ can be obtained in the same manner by restricting the functions ϕ to the class $C_{00}[X]$ having compact support, while the topology σ results if ϕ is allowed to be any function in $C^*[X]$; this shows that $\kappa \leq \beta \leq \sigma$. Before giving the list of properties of the strict topology which makes it so useful, let us mention some of the characteristics of the topologies σ and κ. In the topology σ, $C^*[X]$ is a complete normed linear space; the subspace $C_{00}[X]$ is not dense, but its closure is the space $C_0[X]$. In the compact-open topology κ, $C^*[X]$ is a locally convex linear space. If X is not compact, then κ is not a normable topology; it is metrizable only if X is σ-compact. The subspace $C_{00}[X]$ is κ-dense in $C^*[X]$; in fact, it is κ-dense in the space $C[X]$. The topology is not complete; the completion of $\langle C^*[X], \kappa\rangle$ is the complete space $\langle C[X], \kappa\rangle$.

THEOREM 1. *Let* X *be locally compact,* E *locally convex and complete, and let* β *be the strict topology on* $C^*[X: E]$. *Then*

(i) $\kappa \leq \beta \leq \sigma$, *with equality only if* X *is compact;*

(ii) *the space* $C^*[X: E]$ *is topologically complete;*

(iii) *the topologies* β *and* σ *have the same bounded sets;*

(iv) *on any* β*-bounded set in* $C^*[X: E]$, *the topology* β *agrees with the compact-open topology* κ;

(v) *a sequence* $\{f_n\}$ *in* $C^*[X: E]$ *is* β*-convergent if and only if it is* σ*-bounded and* κ*-convergent;*

(vi) *the subspace* $C_{00}[X: E]$ *is* β*-dense in* $C^*[X: E]$;

(vii) *the topology* β *is metrizable only if* X *is compact.*

Of this list, items (iii), (iv) and (v) were proved in [2] for the function algebra $C^*[X]$ in the case when X is a locally compact group, and these proofs may be carried over to the present context with little change. We prove the remaining items separately.

Proof of (ii). Let $\{f_\alpha\}$ be a net which is β-Cauchy. Since $\kappa < \beta$, $\{f_\alpha\}$ is then κ-Cauchy, and therefore converges in the topology κ to a mapping $F \in C[X: E]$. For any $\phi \in C_0[X]$, $\phi f_\alpha \xrightarrow{\kappa} \phi F$; since $\{f_\alpha\}$ is β-Cauchy, $\{\phi f_\alpha\}$ is σ-Cauchy, and thus converges in the topology σ to a mapping H in $C^*[X: E]$. We must therefore have $H(x) = \phi(x) F(x)$ for all $x \in X$. Since each ϕf_α is in $C_0[X: E]$, which is σ closed, we have $\phi F \in C_0[X: E]$. Moreover, this is true for each $\phi \in C_0[X]$, so that we can apply Lemma 4 and conclude that $F \in C^*[X: E]$. Since $\phi f_\alpha \xrightarrow{\sigma} \phi F$ for each ϕ, we have $f_\alpha \xrightarrow{\beta} F$, and β is a complete topology.

Proof of (vi). If K is a compact set in X, let ψ_K be any function with compact support, taking values in [0, 1], and with $\psi_K(x) = 1$ for all $x \in K$. If f is any mapping in $C^*[X: E]$, then $f_K = \psi_K f$ belongs to $C_{00}[X: E]$. Moreover, the collection of functions $\{f_K\}$ is σ-bounded, and if K_1 is a compact set contained in K, then $\|f_K - f\|_{K_1, \nu} = 0$. If we order the subscripts by inclusion in the obvious way, $\{f_K\}$ becomes a σ-bounded net which is κ-convergent to f. By virtue of (iv), this shows that a particular net of mappings in $C_{00}[X: E]$ has been found which is β-convergent to f, and this subspace is β-dense in $C^*[X: E]$.

Proof of (i). When X is not compact, the topologies κ, β, and σ are all distinct; for, κ is not complete, and $C_{00}[X: E]$ is not σ-dense in $C^*[X: E]$.

Proof of (vii). If β were metrizable, then β and σ would be two complete metrizable topologies on the same linear space, with $\beta \leq \sigma$. By a standard deduction from the open mapping theorem, $\beta = \sigma$; but, this can happen only when X is compact.

4. THE DUAL SPACE OF $C^*[X]$

If X is a compact space, and if we use the uniform topology σ, then the dual space of $C^*[X]$ if $\mathfrak{M}[X]$, the space of bounded Radon measures on X. However, when X is only locally compact, the uniform dual of $C^*[X]$ is obtained as the space of Radon measures on βX. If $\mu \in \mathfrak{M}[\beta X]$, and $f \in C^*[X]$, then the functional defined by μ is represented by

$$L(f) = \int_{\beta X} f^* \, d\mu ,$$

where f^* is the unique extension of f to the compact space βX. If this is thrown back on X, then μ corresponds to only a finitely additive measure on X, in general. Thus, the uniform dual of $C^*[X]$ is somewhat unwieldy. The situation is better for the subalgebra $C_0[X]$; as observed earlier, this can be replaced by a closed maximal ideal in the space $C[X^0]$, where X^0 is the one-point compactification of X. The uniform dual of $C_0[X]$ (and thus also for $C_{00}[X]$) turns out to be $\mathfrak{M}[X]$ itself. Returning to the larger space $C^*[X]$, but using the compact-open topology κ instead, we find the dual space to be $\mathfrak{M}_0[X]$, the space of measures with compact support [9]. It is thus of interest that the *strict* dual of $C^*[X]$ is exactly $\mathfrak{M}[X]$ itself.

LEMMA 5. *Each measure μ in $\mathfrak{M}[X]$ has a σ-compact support.*

Let μ be a positive measure on X, and let L be the corresponding linear functional defined on $C_{00}[X]$ by

$$L(f) = \int_X f \, d\mu . \tag{1}$$

Since μ is bounded, L is σ-continuous, and there exists a number $M = \|L\|$ such $|L(f)| \leq M\|f\|$ for all $f \in C_{00}[X]$. Choose $f_n \in C_{00}[X]$ with $\|f_n\| = 1$ and $\lim L(f_n) = M$. Let K_n be the compact support of f_n, and let $S_\mu = \bigcup K_n$. We show that S_μ contains the support of μ. Let $g \in C_{00}[X]$ be any function which is zero on S_μ. We can assume that $\|g\| < 1/2$. Then, $\|f_n + g\| = \|f_n - g\| = 1$ for each n, so that

$$\|L(f_n \pm g)\| = \|L(f_n) \pm L(g)\| \leq M .$$

Letting n increase, we see that $L(g) = 0$. As a consequence of this argument, we see that (1) can be replaced by

$$L(f) = \int_{S_\mu} f \, d\mu \quad (f \in C_{00}[X]). \tag{2}$$

Note also that $M = \mu(X) = \mu(S_\mu)$.

THEOREM 2. *Every strictly continuous linear functional on* $C^*[X]$ *has the representation* (2), *so that the strict dual of* $C^*[X]$ *is* $\mathfrak{M}[X]$.

Let L be any positive β-continuous linear functional on $C^*[X]$. Since $\beta \leq \sigma$, L is σ-continuous on the subspace $C_{00}[X]$. By the foregoing remarks, L has there the representation (2). Since $C_{00}[X]$ is β-dense in $C^*[X]$, it is only necessary to show that (2) defines a β-continuous functional on $C_{00}[X]$. By Lemma 5, we can write $S_\mu = \bigcup K_n$. Put $a_n = \mu(K_n - K_{n-1})$, so that $a_n \geq 0$ and $\Sigma a_n = M$. Choose a sequence

$\{b_n\}$ with $b_n \downarrow 0$, and with $\Sigma\, a_n/b_n < 2M$. Set $c_n = b_n - b_{n+1}$, so that

$$b_n = c_n + c_{n+1} + \cdots .$$

Then, construct functions $\psi_n \in C_{00}[X]$ taking values in $[0, 1]$, and such that $\psi_n(x) = 1$ for $x \in K_n$ while $\psi_n(x) = 0$ for $x \notin K_{n+1}$. Construct a function ψ by setting $\psi(x) = \Sigma\, c_n \psi_n(x)$. Then, it is easily seen that $\psi \in C_0[X]$, that ψ is zero except on S_μ, and that $\psi(x) \geq b_n$ for $x \in K_n - K_{n-1}$. The function $1/\psi$ is then continuous on S_μ, and integrable (μ), with $\int (1/\psi)\, d\mu < 2M$. Let $f \in C_{00}[X]$ with $\|f\|_\psi < \varepsilon/2M$. Then

$$\left|\int_X f\, d\mu\right| \leq \frac{\varepsilon}{2M} \int_{S_\mu} \frac{1}{\psi}\, d\mu ,$$

which proves that (2) is β-continuous.

Several remarks are in order. The strict topology on $C^*[X]$ is not the only one which yields $\mathfrak{M}[X]$ for the dual space, for this is also true for any topology τ which lies between the weak paired topology $w(C^*, \mathfrak{M})$ and the Mackey topology $m(C^*, \mathfrak{M})$. However, this remark is not too enlightening, since these special topologies are difficult to describe directly in terms of $C^*[X]$. In particular, the following question can be raised: *is it in fact true that the strict topology* β *coincides with the Mackey topology* m?

Other problems remain. It is easily seen that the space $\mathfrak{M}[X]$ can also be regarded as the space of linear functionals on $C^*[X]$ that are β-continuous on β-bounded sets [2]. It would therefore be of interest to have a similar representation theorem for the space of linear transformations from $C^*[X: E]$ into E that are β-continuous on β-bounded sets, or even for the strict dual of $C^*[X: E]$ itself.

5. A STONE-WEIERSTRASS THEOREM FOR C*[X]

When X is compact, a uniformly closed subalgebra of $C[X]$ which separates points must be $C[X]$ itself. This fails for $C^*[X]$ when X is not compact, since there are proper (closed) maximal ideals in $C^*[X]$ arising from points in $\beta X - X$ [10], [3].

THEOREM 3. *Let* X *be locally compact, and let* $\mathfrak{A}$ *be a* β*-closed subalgebra of* $C^*[X]$. *If* $\mathfrak{A}$ *separates points of* X, *and contains a function vanishing nowhere, then* $\mathfrak{A} = C^*[X]$.

Let $g \in \mathfrak{A}$ and let it be everywhere positive. We show first that $\mathfrak{A}$ contains the function 1. By the usual series argument (see [10]), $g^r \in \mathfrak{A}$ for each $r > 0$. In particular, $h_n = g^{1/n}$ is in $\mathfrak{A}$ for $n = 1, 2, \cdots$. Since $g(x) > 0$ for every $x \in X$, and $\|h_n\| \leq \|g\|$, the sequence $\{h_n\}$ is uniformly bounded in X, and converges to 1, uniformly on each compact set. Since $\mathfrak{A}$ is β-closed, $1 \in \mathfrak{A}$. By the usual argument, $\mathfrak{A}$ contains $|f|$ whenever it contains f, so that $\mathfrak{A}$ is a sublattice of $C^*[X]$ (see [10]). Let F be any function in $C^*[X]$, and set $M = \|F\|$. Given any $\phi \in C_0[X]$ with $\|\phi\| \leq 1$, and any $\varepsilon > 0$, choose a compact set K such that $|\phi(x)| < \varepsilon$ for any x not in K. Let $\mathfrak{A}_K$ be the set of functions on K obtained by restricting each $f \in \mathfrak{A}$ to the set K. $\mathfrak{A}_K$ is a separating subalgebra of $C[K]$. By the standard Stone-Weierstrass theorem, $\mathfrak{A}_K$ is σ-dense in $C[K]$, so that there exists a function $f \in \mathfrak{A}$ such that $|f(x) - F(x)| < \varepsilon$ for all $x \in K$. Let f_0 be the function $\{f \cap 2M\} \cup -2M$. This is in $\mathfrak{A}$, since $\mathfrak{A}$ is a lattice. It is easily seen that $|f_0(x) - F(x)| < \varepsilon$ for $x \in K$, and that for each $x \in X$,

$$|(f_0(x) - F(x))\phi(x)| \leq \sup_{x\in K} |f_0(x) - F(x)||\phi(x)| + \sup_{x\notin K} |f_0(x) - F(x)||\phi(x)|$$

$$< \varepsilon + 3M\varepsilon.$$

Thus, any β-neighborhood of F contains a function in $\mathfrak{A}$; since $\mathfrak{A}$ is β-closed, $F \in \mathfrak{A}$, and $\mathfrak{A} = C^*[X]$.

If X is slightly specialized, the extra condition on $\mathfrak{A}$ can be removed:

THEOREM 4. *Let* X *be locally compact and* σ-*compact, and let* $\mathfrak{A}$ *be a subset of* $C^*[X]$ *which separates points of* X. *Then the algebraic closure of* A *in* $C^*[X]$ *is* β-*dense.*

Replace $\mathfrak{A}$ by the β-closed subalgebra of $C^*[X]$ which it generates. By Theorem 3, it is only necessary to show that $\mathfrak{A}$ contains a function that is everywhere positive. Let $X = \bigcup K_n$ with K_n compact in X. By the standard Stone-Weierstrass theorem, $\mathfrak{A}_{K_n}$ is uniformly dense in $C[K_n]$. Choose $f_n \in \mathfrak{A}$ with $f_n(x) \geq 1$ for all $x \in K_n$. Let $\|f_n\| = c_n$ and set $g_n = (n c_n)^{-2}(f_n)^2$. Then $g_n \in \mathfrak{A}$, and is nonnegative on X, strictly positive on K_n, and $\|g_n\| \leq 1/n^2$. Set $g = \Sigma g_n$. This is uniformly (and hence strictly) convergent, so that $g \in A$. Clearly, g is strictly positive on X.

A slightly weaker form of Theorem 3 was obtained independently by Karel de Leeuw. We have not succeeded in removing the extra hypothesis on A, in the general case; it is certainly not necessary, as is shown by the β-dense subalgebra $C_{00}[X]$ which contains *no* nonvanishing function. One additional remark may be made: all proper maximal ideals in $C^*[X]$ are uniformly closed, including those that arise as annihilators of points in $\beta X - X$. However, these 'free' ideals contain $C_{00}[X]$, so that they are in fact dense in the strict topology. This proves the following interesting fact.

COROLLARY 1. *If* X *is a locally compact space, then there is a natural one-to-one correspondence between the points of* X *and the strictly closed proper maximal ideals in* $C^*[X]$.

COROLLARY 2. *If* X *and* Y *are locally compact spaces, and* $C^*[X]$ *and* $C^*[Y]$ *are algebraically and topologically isomorphic, in the strict topologies, then* X *and* Y *are homeomorphic.*

6. A STONE-WEIERSTRASS THEOREM FOR C*[X: E]

If the linear space E is also an algebra, then the space $C^*[X: E]$ acquires a natural algebraic structure, and it is then possible to seek an analogue of the approximation theorem. The strongest results in this direction are those of Kaplansky [5] and Yood [12]; in these, X is a compact space, and E is either an algebra of operators on Hilbert space, or a commutative Banach algebra. A special case of the latter occurs if E itself is the function algebra C[Y] for a compact space Y; in this case, $C^*[X: E]$ is exactly $C[X\times Y]$.

However, when E is only a linear space, the only natural algebraic structure which $C^*[X: E]$ possesses is that of a $C^*[X]$ module. It is therefore reasonable to conjecture a theorem of the following sort: if $\mathfrak{A}$ is a closed submodule of $C^*[X: E]$

which in addition . . . . , then $\mathfrak{A} = C^*[X: E]$. The blank is to be filled in with the appropriate replacement for "separates points of X". It is clear that "separates points of X" would not be enough; for if V is a proper closed subspace of E, then $\mathfrak{A}_{x_0} = \{\text{all } f \in C^*[X: E] \text{ with } f(x_0) \in V\}$ is a proper closed submodule of $C^*[X: E]$. The theorem which follows is a step in the direction of such a general theorem.

THEOREM 5. *Let* X *be locally compact and metrizable, and* E *finite-dimensional, and let* $\mathfrak{A}$ *be a* β*-closed submodule of* $C^*[X: E]$. *Then, if* $\mathfrak{A}(x) = E$ *for each* $x \in X$, $\mathfrak{A}$ *is identical with* $C^*[X: E]$.

The assumptions on $\mathfrak{A}$ are these: (i) $\mathfrak{A}$ is a closed subspace of $C^*[X: E]$; (ii) if $f \in \mathfrak{A}$ and $\phi \in C^*[X]$, then $\phi f \in \mathfrak{A}$; (iii) given any $x \in X$ and $p \in E$, there is an $f \in \mathfrak{A}$ with $f(x) = p$. Let F be any mapping in $C^*[X: E]$. We shall prove that F lies in $\mathfrak{A}$ by showing first that F can be locally matched within $\mathfrak{A}$. Let x_0 be any point in X, and let $\theta_1, \theta_2, \cdots, \theta_n$ be a basis for E. Choose $g_1, \cdots, g_n$ in $\mathfrak{A}$ with $g_j(x_0) = \theta_j$. Then, there exists a closed neighborhood G about x_0 in which $g_1(x), \cdots, g_n(x)$ are independent. In fact, if $L_1, \cdots, L_n$ are functionals on E orthogonal to the θ_j then $g_j(x) = \Sigma_1^n L_k(g_j(x))\, \theta_k$; setting

$$D(x) = \begin{bmatrix} L_1(g_1(x)) & \cdots & L_n(g_1(x)) \\ \cdots & \cdots & \cdots \\ L_1(g_n(x)) & \cdots & L_n(g_n(x)) \end{bmatrix}$$

we have a matrix-valued function with continuous entries, with $D(x_0)$ the identity matrix. There exists then a closed neighborhood G about x_0 in which $D(x)$ is nonsingular and has a nonsingular inverse $C(x) = [c_{kj}(x)]$ whose entries are also continuous in G. Accordingly, for each $x \in G$,

$$\theta_k = \sum_{j=1}^{n} c_{kj}(x)\, g_j(x) \qquad (k = 1, 2, \cdots, n)$$

and the statements

$$F(x) = \sum_{k=1}^{n} L_k(F(x))\, \theta_k = \sum_{k=1}^{n} \sum_{j=1}^{n} L_k(F(x))\, c_{kj}(x) g_j(x) = \sum_{1}^{n} \phi_j(x)\, g_j(x)$$

hold for each $x \in G$. The functions ϕ_j are continuous on G. Extend each to all of X in such a way that it is bounded and continuous. Then, we have found a function $g = \Sigma_1^n \phi_j g_j$ in $\mathfrak{A}$ such that $g(x) = F(x)$ on a neighborhood of x_0.

The remainder of the proof is more or less standard. The interiors of the sets G cover X. From them, we can extract a locally finite subcovering $\{G_j\}$ and construct the associated partition of unity, ψ_j, with $\psi_j(x) \geq 0$ for all x, while $\psi_j(x) = 0$ if x is outside of G_j, and $\Sigma \psi_j(x) = 1$ for all x. By the previous construction, we can find a mapping $g_j \in \mathfrak{A}$ which agrees with F on G_j. Construct a mapping $f_N \in \mathfrak{A}$ by

$$f_N(x) = \sum_{1}^{N} \psi_j(x)\, g_j(x) .$$

Then, it is easily seen that $\{f_N\}$ is uniformly bounded on X, and converges to F, uniformly on compact sets in X; since this implies β-convergence, and since $\mathfrak{A}$ is closed, $F \in \mathfrak{A}$.

If the space X is in fact compact, a result analogous to Theorem 5 can be obtained with E quite general. For completeness, we insert a sketch of this and its proof. Let E be any locally convex linear space, let X be compact, and suppose that $\mathfrak{A}$ is a σ-closed submodule of C[X: E]. Then, $\mathfrak{A}$ again equals C[X: E] if it is true that $\mathfrak{A}(x) = E$ for each $x \in X$. Let W be a preassigned convex neighborhood of the origin in E, and let $F \in C[X: E]$. For any $x_0 \in X$, we can choose $f_{x_0} \in \mathfrak{A}$ so that $f_{x_0}(x_0) = F(x_0)$, and then choose a neighborhood U_{x_0} about x_0 such that

$$f_{x_0}(x) - F(x) \in W$$

for all $x \in U_{x_0}$. The open sets U_{x_0} cover X; since X is compact, a finite number of these will suffice. Denoting these by $U_1, U_2, \cdots, U_n$, and the corresponding functions f_{x_0} by $f_1, f_2, \cdots, f_n$, we choose functions $\psi_j \in C[X]$ such that $1 = \Sigma\psi_j(x)$ for all $x \in X$ and with $\psi_j(x) = 0$ for x outside U_j and $\psi_j(x) \geq 0$ for $x \in U_j$. It is then easily verified that the function given by $f = \Sigma_1^n \psi_j f_j$ is in $\mathfrak{A}$, and that $f(x) - F(x) \in W$ for all $x \in X$. Since this holds for each W, and since $\mathfrak{A}$ is closed, $F \in \mathfrak{A}$.

The 'separating' condition that we have used in these two theorems is a natural one. It is equivalent to the assertion that if $x_0 \in X$ and V is a proper closed subspace of E, then there exists a mapping f in $\mathfrak{A}$ with $f(x_0) \notin V$. Note that this condition is violated by the closed submodule $\mathfrak{A}_{x_0}$ introduced above; these maximal submodules might be expected to play the role of maximal ideals in the study of the structure of a general closed submodule. Another point of view is also suggestive. Assume that E is a normed space, and let S be the unit ball in the dual space E'. If $f \in C^*[X: E]$, let f^* be the real-valued function defined on $S \times X$ by

$$f^*(L, x) = L(f(x)).$$

The mapping $f \to f^*$ is an embedding of $C^*[X: E]$ into $C^*[S \times X]$, and a submodule $\mathfrak{A}$ of $C^*[X: E]$ obeys the separating condition of Theorem 5 if and only if its image in $C^*[S \times X]$ separates points in the usual sense.

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