TWO APPLICATIONS OF CLOSE-TO-CONVEX FUNCTIONS

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1. In this note we shall extend two recent results on univalent functions. One result concerns the univalence, near $z = \infty$, of certain functions considered by L. Tchakaloff [5]; the other result concerns a domain of univalence of the function $\int_0^z e^{-\zeta^2} d\zeta$, considered by V. S. Rogozhin [4]. We shall make use of the close-to-convex functions introduced by W. Kaplan [2] and Umezawa [6].

2. The following result gives slightly more precise information concerning the domain of univalence of certain functions considered by Tchakaloff [5].

THEOREM 1. *Let* $a_1, a_2, \cdots, a_n$ *be distinct points contained in the disc* $|z - z_0| < R$, *and let* $A_1, A_2, \cdots, A_n$ *be positive constants. Then the function*

$$f(z) \equiv \sum_{k=1}^{n} \frac{A_k}{z - a_k} \tag{1}$$

is univalent in a star-shaped neighborhood of $z = \infty$ *that contains the exterior of the circle* $|z - z_0| = R\sqrt{2}$. *Moreover, the function*

$$g(\zeta) \equiv f\left(\frac{R^2}{\zeta - z_0} + z_0\right) \tag{2}$$

is univalent in a convex domain containing the disc $|\zeta - z_0| < R/\sqrt{2}$.

Proof. We adapt Tchakaloff's proof. If ζ_1 and ζ_2 are distinct points, then (1) and (2) yield the relation

$$g(\zeta_2) - g(\zeta_1) = \frac{(\zeta_2 - \zeta_1)}{R^2} \sum_{k=1}^{n} \frac{A_k}{h_k(\zeta_1, \zeta_2)},$$

where

$$h_k(\zeta_1, \zeta_2) \equiv \left(1 - \frac{(\zeta_2 - z_0)(a_k - z_0)}{R^2}\right)\left(1 - \frac{(\zeta_1 - z_0)(a_k - z_0)}{R^2}\right).$$

Therefore

$$|g(\zeta_2) - g(\zeta_1)| \geq \frac{|\zeta_2 - \zeta_1|}{R^2} \sum_{k=1}^{n} A_k \Re \frac{1}{h_k(\zeta_1, \zeta_2)}, \tag{3}$$

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from which it follows that $g(\zeta)$ is univalent on each set E with the property that the inequalities $\Re h_k(\zeta_1, \zeta_2) > 0$ $(k = 1, 2, \cdots, n)$ hold for all pairs of distinct points ζ_1 and ζ_2 in E. Now let E be a maximal set with this property, and let E_k be the subset of E consisting of two quadrants formed by two mutually perpendicular lines through the point $[R^2/(a_k - z_0)] + z_0$ and symmetric about the line through the points z_0 and $[R^2/(a_k - z_0)] + z_0$. One of these quadrants, say D_k, contains the disc $|\zeta - z_0| < R/\sqrt{2}$.

Now let $D \equiv \bigcap_1^n D_k$. Then the right-hand member of (3) is positive for each pair of distinct points ζ_1, ζ_2 in D. From (3) it follows that $g(\zeta)$ is univalent in D. Since each D_k is convex and contains the disc $|\zeta - z_0| < R/\sqrt{2}$, it follows that D is convex and contains the same disc. The rest of the announced result now follows. This completes the proof.

Remark 1. We note that the set E_k is precisely the set where

$$\Re \left\{ \frac{d}{d\zeta} \left(\frac{R^2}{\zeta - z_0} + z_0 - a_k \right)^{-1} \right\} > 0 .$$

Hence D is a set where $\Re g'(\zeta) > 0$. Since D is convex, it follows from the Noshiro-Warschawski theorem that $g(\zeta)$ is univalent in D. It now follows from Kaplan's fundamental result that the image of D under $g(\zeta)$ is a close-to-convex domain.

Remark 2. By the same methods we can extend Tchakaloff's results concerning functions of the form

$$f_1(z) \equiv \frac{1}{\pi} \int_0^{2\pi} \frac{d\alpha(t)}{z - e^{it}}$$

and

$$f_2(z) \equiv \int_{-1}^{1} \frac{d\alpha(t)}{z - t},$$

where $\alpha(t)$ is a nonconstant monotone function. For example, we can show that $g_1(\zeta) \equiv f_1(1/\zeta)$ is close-to-convex for $|\zeta| < \sqrt{2}/2$, and we can show that $g_2(\zeta) \equiv f_2(1/\zeta)$ is close-to-convex for $|\xi| + |\eta| < 1$, where $\zeta = \xi + i\eta$. These last results yield corresponding results for $f_1(z)$ and $f_2(z)$, respectively.

If we make the normalization $\alpha(2\pi) - \alpha(0) = 2\pi$, then we can write

$$g_1(\zeta) \equiv f_1(1/\zeta) = \zeta + \frac{\zeta}{2\pi} \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d\alpha(t) . \tag{4}$$

From (4) we easily obtain a characterization of $g_1(\zeta)$ of the form $\Re[g_1(\zeta)/\zeta - 1] > 0$ for $|\zeta| < 1$. Furthermore, it follows from (4) that the $g_1(\zeta)$ have a close connection with certain star maps studied by Kaplan [1; p. 11]. This connection will be studied elsewhere.

Remark 3. It would be interesting to obtain geometric characterizations of the maps $f_1(z)$, $f_2(z)$, $g_1(\zeta)$, $g_2(\zeta)$ defined above, similar to the geometric characterization of convex, star and close-to-convex maps.

3. In a recent note, Rogozhin proved that the function

$$\Phi(z) \equiv \int_0^z e^{-\zeta^2} d\zeta \qquad (z = x + iy) \tag{5}$$

is univalent in any convex subdomain of the simply connected domain bounded by the hyperbolas $xy = \pm\pi/4$ [4]. We shall use a theorem of Umezawa to extend this result.

THEOREM 2. *The function* $\Phi(z)$ *in* (5) *is univalent in the simply connected domain* G *bounded by the hyperbolas* $xy = \pm\pi/4$, *and it maps* G *onto a close-to-convex domain.*

Proof. For any directed analytic arc C, let $\triangle C$ denote the change in inclination of the unit tangent vector to C.

Let Γ denote the oriented boundary of G. It is geometrically clear that

$$\int_C d[\arg d\Phi(z)] = \triangle C + \int_C d[\arg \Phi'(z)] \geq -\pi$$

holds for any arc C on Γ. It is also clear, from the shape of Γ and from the relation $\arg \Phi'(z) = -2xy$, that we can obtain a simple closed analytic curve Γ' in G, close to Γ, such that $\triangle\Gamma' = 2\pi$ and such that

$$\int_C d[\arg d\Phi(z)] = \triangle C + \int_C d[\arg \Phi'(z)] > -\pi$$

holds for all arcs C on Γ'. Therefore we may apply a theorem due to Umezawa [6; p. 173] to conclude that $\Phi(z)$ is univalent inside Γ'. Since Γ' can be chosen as close to Γ as we please, it follows that $\Phi(z)$ is univalent in G. It also follows from a recent result due to the present author that $\Phi(z)$ maps each Γ' onto a simple, closed, close-to-convex curve [3]. Hence $\Phi(z)$ maps G onto a close-to-convex domain.

Remark 4. Professor J. L. Ullman has remarked that the function

$$w = \sqrt{\tanh \frac{1}{2} z^2} = z + \cdots$$

maps G conformally onto $|w| < 1$, and that hence the inverse images of the circles $|w| = r < 1$ can serve as the approximating curves Γ' above.

Remark 5. It would be interesting to obtain a larger domain of univalence for $\Phi(z)$ [4].

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