LOCALLY CONVEX TOPOLOGICAL VECTOR LATTICES AND THEIR REPRESENTATIONS

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INTRODUCTION

A Banach lattice is defined as a lattice-ordered Banach space, with real scalars, in which the ordering and the norm are related by the postulate

$$\text{(A)} \qquad |x| \leq |y| \quad \text{implies} \quad ||x|| \leq ||y||, \qquad \text{where } |x| = x \cup (-x).$$

It is the purpose of this paper to discuss a generalization of the Banach lattice, namely, an object which will be called a locally convex vector lattice, and to examine some of its properties. We also define the locally m-convex vector lattice and the locally l-convex vector lattice, corresponding to the abstract M-space and the abstract L-space, two specializations of the Banach lattice that are due to Kakutani [3, 4]. By means of Kakutani's concrete representation theorems for these spaces, and by the use of the projective limit, it is possible to obtain representation theorems on the locally convex vector lattices.

1. PRELIMINARY REMARKS ON VECTOR LATTICES AND BANACH LATTICES

For the basic definitions and properties of vector lattices, the reader is referred to [1].

DEFINITION 1.1. An *ideal* I in the vector lattice E is a linear subspace of E, with the property that if $a \in I$ and $|x| \leq |a|$, then $x \in I$. (Equivalently, if $a, b \in I$ and if $a \cap b \leq x \leq a \cup b$, then $x \in I$.)

LEMMA 1.1. *If* B *is a Banach lattice, and if* I *is a closed ideal in* B, *then* B/I *is a Banach lattice.*

Proof. We denote the elements of B/I by letters (or numbers) with bars; B/I is a Banach space under the norm $||z|| = \inf_{t \in \bar{z}} ||t||$. Showing that the relation $|\bar{x}| \leq |\bar{y}|$ implies $||\bar{x}|| \leq ||\bar{y}||$ is equivalent to showing that (1) if $\bar{0} \leq \bar{x} \leq \bar{y}$, then $||\bar{x}|| \leq ||\bar{y}||$ and (2) $||\bar{x}|| = ||\,|\bar{x}|\,||$ for all $\bar{x} \in B/I$.

To prove (1): If $\bar{0} \leq \bar{x}$ and $t \in \bar{x}$, then $|t| \in \bar{x}$. Hence, for any $\bar{x} \geq \bar{0}$, we have $||\bar{x}|| = \inf_{0 \leq t \in \bar{x}} ||t||$. In $\bar{y}$, choose any $t \geq 0$, and in $\bar{x}$, choose any $w \geq 0$. Then $\overline{w \cap t} = \bar{w} \cap \bar{t} = \bar{x} \cap \bar{y} = \bar{x}$, since $\bar{x} \leq \bar{y}$. Let $z = w \cap t$; then $z \in \bar{x}$, and $0 \leq z \leq t$. Hence, $||z|| \leq ||t||$ and $||\bar{x}|| = \inf_{0 \leq z \in \bar{x}} ||z|| \leq \inf_{0 \leq t \in \bar{y}} ||t|| = ||\bar{y}||$.

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To prove (2): If $z \in \bar{x}$, then $|z| \in \overline{|x|}$; hence, for every element in $\bar{x}$, there is one of the same norm in $\overline{|x|}$. Therefore

$$||\bar{x}|| = \inf_{t \in \bar{x}} ||t|| \geq \inf_{w \in |\bar{x}|} ||w|| = ||\overline{|x|}|| = |||\bar{x}|||.$$

To demonstrate this inequality in the other direction, it suffices to show the following: if $0 \leq t \in \overline{|x|}$, then there exists a $w \in \bar{x}$ such that $|w| \leq t$. For we would then have $||w|| \leq ||t||$, and $||\bar{x}|| = \inf_{w \in \bar{x}} ||w|| \leq \inf_{0 \leq t \in \overline{|x|}} ||t|| = ||\overline{|x|}|| = |||\bar{x}|||$. It can be verified that the element $(x \cup 0) \cap t - [(-x) \cup 0] \cap t$ satisfies these requirements for w.

Corollary. If $\bar{0} \leq \bar{x}$, *then* $||\bar{x}|| = \inf_{0 \leq t \in \bar{x}} ||t||$.

An *abstract* M-*space* is a Banach lattice with the property M: If $x \cap y \geq 0$, then $||x \cup y|| = \max[||x||, ||y||]$. In a Banach lattice B, property M is equivalent to the assertion that the unit sphere is closed under the lattice operation $\cup$, or, by duality, under $\cap$; this is easily verified. An element e in a Banach lattice is called a *strong unit* if $e > 0$, $||e|| = 1$, and if $||x|| \leq 1$ implies $x \leq e$. A strong unit is unique, and its existence implies property M. A subset S of a Banach lattice B is *order-bounded* if there exist elements a and b in B such that $a \leq s \leq b$ for all $s \in S$. An order-bounded set is automatically metrically bounded, but the converse is true if and only if B can be provided with an equivalent norm under which it has a strong unit e. An *abstract* L-*space* is a Banach lattice with property L: If $x \cap y \geq 0$, then $||x + y|| = ||x|| + ||y||$. It is assumed that the reader is familiar with Kakutani's representation theorems and with the nature of the closed ideals in the concrete representations.

2. LOCALLY CONVEX VECTOR LATTICES

DEFINITION 2.1. A locally convex, linear Hausdorff space E which is also a vector lattice will be called a *locally convex vector lattice* if the seminorms $\{p_\alpha\}_{\alpha \in J}$ corresponding to some fundamental system of neighborhoods $\{V_\alpha\}_{\alpha \in J}$ all satisfy the condition

(a) if $|x| \leq |y|$, then $p_\alpha(x) \leq p_\alpha(y)$.

The condition (a) guarantees the continuity of the lattice operations in E, just as postulate (A) performs that service in Banach lattices. Whenever a fundamental system of neighborhoods is referred to, it will be understood to satisfy (a).

DEFINITION 2.2. A vector lattice E is said to be *Archimedean* if the condition that $u > 0$ and $x \leq \lambda u$, for all scalars $\lambda > 0$, implies that $x \leq 0$.

LEMMA 2.1. *A locally convex vector lattice is Archimedean.*

Proof. If, for some $u > 0$, it is true that $x \leq \lambda u$ for all $\lambda > 0$, then

$$0 \leq x \cup 0 \leq \lambda u \cup 0 = \lambda u,$$

and $p_\alpha(x \cup 0) \leq p_\alpha(\lambda u) = \lambda p_\alpha(u)$ for all $\lambda > 0$. Hence, $p_\alpha(x \cup 0) = 0$ for all α. Since E is a Hausdorff space, we have $x \cup 0 = 0$ and $x \leq 0$.

Examples of locally convex vector lattices.

(1) Normed vector lattices. Vector sublattices of locally convex vector lattices in the subspace topology. Cartesian products of locally convex vector lattices in the product space topology. Quotients of locally convex vector lattices by closed ideals, in the quotient space topology.

(2) The set of all real-valued continuous functions on a completely regular topological space. In the compact-open topology, this locally convex vector lattice is complete.

(3) Let μ be a Radon measure on a locally compact Hausdorff space S, and let $\mathfrak{L}^1(S)$ be the set of real-valued locally integrable functions on S. The sets of the form $V(K,\lambda) = \{x(t) \mid \int_K x(t)\, d\mu < \lambda\}$, where K is a compact subset of S and $\lambda > 0$, are a base for a complete locally convex topology on $\mathfrak{L}^1(S)$.

Let E be a locally convex vector lattice; some notations and remarks follow:

$\{V_\alpha\}_{\alpha \in J}$ denotes a fundamental system of neighborhoods of 0 in E.

$\{p_\alpha\}_{\alpha \in J}$ denotes the corresponding set of seminorms.

$N_\alpha = \{x \mid p_\alpha(x) = 0\}$; N_α is an ideal in E.

$E_\alpha = E/N_\alpha$; E_α is a normed vector lattice.

$\bar{p}_\alpha$ denotes the norm on E_α; $\bar{p}_\alpha(\bar{x}) = p_\alpha(x)$ for any $x \in \bar{x}$.

π_α stands for the natural lattice homomorphism of E onto E_α; it is both continuous and open.

x_α designates $\pi_\alpha(x)$, the component of x in E_α.

If E is complete, then the E_α are Banach lattices.

LEMMA 2.2. *A topological vector lattice is a locally convex vector lattice if and only if it is isomorphic to a vector sublattice of a Cartesian product of normed vector lattices. A complete, locally convex vector lattice is the projective limit of the Banach lattices* $\{E_\alpha\}_{\alpha \in J}$.

The proof of this lemma follows immediately from the definition of the projective limit [8]. An ordering of J is obtained by setting $\alpha \leq \beta$ if and only if $V_\beta \subseteq V_\alpha$; for $\alpha \leq \beta$, let $\pi_{\alpha\beta}$ be the canonical homomorphism which maps E_β onto E_α. Then, denoting the Cartesian product of the E_α by $P_\alpha E_\alpha$, we have

$$E = \{x \in P_\alpha \; E_\alpha \mid \pi_{\alpha\beta}(x_\beta) = x_\alpha\}.$$

There exist vector lattices with a Hausdorff topology (under which all of the algebraic operations are continuous) that are not locally convex vector lattices. For example, in the Banach space of all continuous functions of bounded variation on the unit interval, where

$$||f|| = \sup_{0 \leq t \leq 1} |f(t)| + (\text{the total variation of } f),$$

and where $f \leq g$ means $f(t) \leq g(t)$ for all t in the unit interval, the lattice operations are continuous, but (A) does not hold. Moreover, there is no equivalent norm under which this space is a Banach lattice.

A theorem of G. Birkhoff [1] implies that in any Banach lattice the topology is uniquely determined by the algebraic structure. This is not true for complete locally convex vector lattices. The space of all continuous functions on the unit interval can be equipped with two inequivalent locally convex topologies: the usual norm topology, and the topology of uniform convergence on the countable, compact subsets of the unit interval.

3. REPRESENTATION OF LOCALLY CONVEX VECTOR LATTICES BY SPACES OF CONTINUOUS FUNCTIONS

Let E be a complete, locally convex vector lattice, and let E^* be the set of all continuous linear functionals on E.

DEFINITION 3.1. An element $x^* \in E^*$ is an h-*functional* if it satisfies the following two conditions: if $x \geq 0$, then $x^*(x) \geq 0$, and if $x \cap y = 0$, then either $x^*(x) = 0$ or $x^*(y) = 0$.

The set H of all h-functionals is easily seen to be identical with the set of all continuous linear lattice homomorphisms of E into R, the set of real numbers. The kernel of any h-functional is a maximal lattice ideal in E.

Henceforth it is also assumed that E satisfies the following condition:

(U) There exists in E an element $e > 0$, and a base $\{V_\alpha\}_{\alpha \in J}$ such that $p_\alpha(e) > 0$ for all α, and such that if $p_\alpha(x) \leq p_\alpha(e)$, then $x_\alpha \leq e_\alpha$.

Clearly, if $p_\alpha(x) \leq p_\alpha(e)$ for all α, then $x \leq e$. The element e is the generalization, to the locally convex case, of the strong unit in the Banach lattice. In fact, $e_\alpha / p_\alpha(e)$ is a strong unit in E_α. The following question (whose analogue for normed lattices is affirmatively answered) is left open. In the special case where order-boundedness and boundedness are equivalent for subsets of E, does E satisfy (U)? (A subset T of E is bounded if, for each V_α, there exists a λ_α such that $\lambda_\alpha T \subseteq V_\alpha$.).

Let $M = \{x^* \in H \mid x^*(e) = 1\}$. The following lemma establishes a one-to-one correspondence between the elements of M and the maximal ideals of E.

LEMMA 3.1. *If* $x^* \in H$ *and* $x^*(e) = 0$, *then* $x^* = 0^*$.

Proof. For any $x^* \in H$, there exists an α such that $|x^*(x)| \leq k p_\alpha(x)$, where k is independent of x. From this it follows that if $x \in N_\alpha$, then $x^*(x) = 0$; hence, there is in E_α^* an element x_α^* defined by the equation $x_\alpha^*[\pi_\alpha(x)] = x^*(x)$. (Since π_α is an onto map, every element of E_α appears as a $\pi_\alpha(x)$, for some x. Moreover, if $\pi_\alpha(x) = \pi_\alpha(y)$, then $x^*(x) = x^*(y)$.) It is easily verified that x_α^* is an h-functional in E_α^*. Now suppose that $x^*(e) = 0$; then $x_\alpha^*(e_\alpha / p_\alpha(e)) = 0$. Since E_α is a Banach lattice with the strong unit $e_\alpha / p_\alpha(e)$, it is, by the representation theorem of Kakutani, the space of all continuous real-valued functions on a compact Hausdorff space S; from this it follows that $x_\alpha^* = 0_\alpha^*$, and therefore that $x^* = 0^*$.

Give E^* the $\sigma(E, E^*)$-topology [2], and let M have the relative topology induced on it; then M is completely regular. Let $C(M)$ be the space of all real-valued continuous functions on M. Define the mapping Φ of E into $C(M)$ in the usual way: $\Phi: x \to x(x^*)$. Clearly, Φ is a vector lattice homomorphism of E into $C(M)$. The following lemma shows that Φ is an isomorphism (into).

LEMMA 3.2. *If* $x^*(x) = 0$ *for all* $x^* \in M$, *then* $x = 0$.

Proof. Let x_α be any component of x, and let z_α^* be any h-functional in E_α^*. The composite map $z_\alpha^* \circ \pi_\alpha$ is in H, and $(z_\alpha^* \circ \pi_\alpha)(x) = 0$. Hence $z_\alpha^*(x_\alpha) = 0$ for any

any h-functional $z_\alpha^* \in E_\alpha^*$. By the nature of E_α, we have $x_\alpha = 0_\alpha$, for any α, and hence $x = 0$.

The following theorem has now been proved.

THEOREM 3.1. *A complete, locally convex vector lattice which satisfies the condition* (U) *is algebraically isomorphic to a vector sublattice of the space of all continuous real-valued functions on a completely regular topological space.*

The topological correspondence is the same as in the locally m-convex algebra representation. (See [6] for the details.) The topology of uniform convergence on the compact, equicontinuous subsets of M is the same as the topology of uniform convergence on the equicontinuous subsets of M (see [2] for the definition of equicontinuity). If this topology is called τ_0, and if τ is the original topology of E, then the following theorem holds.

THEOREM 3.2. *The topology* τ *is finer than the topology* τ_0. *Moreover, if* M *satisfies the condition that every compact subset is equicontinuous, then* τ *is finer than the compact-open topology, and* Φ *is a continuous mapping.* (The condition automatically holds if E is metrizable, or if E is a t-space [2].).

LEMMA 3.3. *Let* S *be a topological space, and let* C(S) *be the vector lattice of all continuous real-valued functions on* S, *in the compact-open topology. If* E *is a vector sublattice of* C(S) *which contains the constant functions, together with enough other functions to separate each pair of points of* S, *then* E *is dense in* C(S).

Proof. Consider the functions of C(S), restricted to a compact subset K of S. Apply the well-known theorem concerning the case where the space is compact [3].

THEOREM 3.3. *If the compact-open topology is put on* C(M), *then* $\Phi(E)$ *is dense in* C(M).

The theorem follows at once from the preceeding lemma.

Finally, if E is the space C(S) of all real-valued continuous functions on a completely regular topological space S, and if the compact-open topology is put on E, then S is homeomorphic to M. The demonstration follows the argument given for the case of a locally m-convex algebra [6].

If the postulate (U) is not satisfied, then the situation is best handled in another way. Let E be a complete, locally convex vector lattice satisfying the following condition:

(M′) there exists a fundamental system of neighborhoods $\{V_\alpha\}_{\alpha \in J}$ such that for all α, if $x \cap y \geq 0$, then $p_\alpha(x \cup y) = \max[p_\alpha(x), p_\alpha(y)]$,

or, equivalently,

(M″) there exists a fundamental system of neighborhoods each of which is closed under the lattice operations.

Such a system of neighborhoods is called an m-*base*, and E is called a *locally* m-*convex vector lattice.*

It is clear from Lemma 2.2 that a complete, locally m-convex vector lattice is the projective limit of the M-spaces $E_\alpha = \hat{C}(S_\alpha)$. Moreover, S_α, a subset of E_α^*, is a compact Hausdorff space for each α, and $\hat{C}(S_\alpha)$ is a closed linear subspace of $C(S_\alpha)$ which consists of all continuous real-valued functions $f(x^*)$ defined on S_α and satisfying a certain set of relations of the form $f(x_1^*) = \lambda f(x_2^*)$. (See [3].) Let S′ be the disjoint union of all of the S_α with the obvious locally compact topology. Let $\hat{C}(S')$ be the set of all real-valued continuous functions f defined on S' which satisfy the following two conditions: (i) when restricted to any of the S_α, f is an element x_α of $E_\alpha = \hat{C}(S_\alpha)$, and (ii) if $\alpha < \beta$, and if x_α and x_β are the corresponding restrictions of f, then $\pi_{\alpha\beta}(x_\beta) = x_\alpha$. Finally, let S be the space obtained from S' by identifying all pairs y_1^* and y_2^* of points of S' which satisfy the relation $f(y_1^*) = f(y_2^*)$ for each f in $\hat{C}(S')$. Then define $\hat{C}(S)$ in terms of $\hat{C}(S')$ in the obvious way. It is clear from the discussion above that E is algebraically isomorphic to a subset of $\hat{C}(S)$. Remarks similar to those which were made in connection with Theorem 3.2 apply to the topology of E and the topologies of $\hat{C}(S)$. Finally, by extending in a direct way some results of M. and S. Krein [5], it can be shown that if $\hat{C}(S)$ is given the compact-open topology, then E is isomorphic to a dense subset of $\hat{C}(S)$.

4. REPRESENTATION OF LOCALLY CONVEX VECTOR LATTICES BY SPACES OF LOCALLY INTEGRABLE FUNCTIONS

Let E be a complete, locally convex vector lattice satisfying the following condition.

(L') There exists a base $\{V_\alpha\}_{\alpha \in J}$ such that if $x \cap y \geq 0$, then $p_\alpha(x + y) = p_\alpha(x) + p_\alpha(y)$ for all α.

Such a base is called an l-*base*, and E is called a *locally* l-*convex vector lattice.* Moreover, it is assumed that E satisfies the following additional condition.

(V) There exists in E an $e > 0$ such that $p_\alpha(e) > 0$ for all α, and if $p_\alpha(x \cap e) = 0$, then $p_\alpha(x) = 0$.

It follows that if $e \cap x = 0$, then $x = 0$; that is, e is a weak unit of E. If a locally l-convex vector lattice does not satisfy (V), then it can be shown, exactly as in [4], that E is a direct sum of locally l-convex vector lattices each of which satisfies (V). The following statements on locally l-convex vector lattices with units are proved exactly as their analogues for normed vector lattices. (The algebraic details are identical; the topological arguments simply replace norm convergence by convergence in the locally convex topology.) Let $B = \{b \in E \mid b \cap (e - b) = 0\}$; then B is a Boolean algebra which is closed in E, and every $x \geq 0$ in E can be written as an integral as follows: $x = \int_0^\infty \lambda \, db(\lambda)$. In this representation, $\{b(\lambda)\}$ is a resolution of the unit e (see [4]). By means of the decomposition $y = y^+ - y^-$, the representation can be extended to all elements of E. It remains to be shown how this abstract representation can be replaced by a concrete one.

The space E is the projective limit of the abstract L-spaces E_α, and hence it can be considered as a subset of the Cartesian product of the E_α's. Since each E_α is isometric and lattice-isomorphic to a space $L^1(S_\alpha)$, where S_α is a totally disconnected, compact Hausdorff space, we can use the following two facts: every metrically bounded set in E_α, which is directed under the lattice ordering, converges metrically; and every order-bounded set in E_α has a supremum and an infimum in E_α.

LEMMA 4.1. *The Boolean algebra* B *is closed under the lattice operation of union.*

Proof. Let $\{b^\nu\}$ be any collection of elements of B; then $0 \leq b^\nu \leq e$ and $0_\alpha \leq b_\alpha^\nu \leq e_\alpha$ for all α. The set $\{b_\alpha^\nu\}$ is an order-bounded subset of B_α, and hence $\bigcup_\nu b_\alpha^\nu$ exists in B_α; call it b_α^0. Then $b^0 = \{\cdots, b_\alpha^0, \cdots\}$ is the supremum of $\{b^\nu\}$; $b^0 = \bigcup_\nu b^\nu$.

In particular, B contains, with every countable set of pairwise disjoint elements, the lattice union of the elements of that set.

In order to define a measure on B, proceed as follows. Let $\{V_\beta\}_{\beta \epsilon J}$ be an l-base with the property that there exists an $\alpha \epsilon$ J such that $\alpha \leq \beta$ for all $\beta \epsilon$ J. For any $\beta \epsilon$ J, define $\lambda_\beta = \sup\{\lambda \mid \lambda V_\beta \subset V_\alpha\}$; it can be assumed that $\lambda_\beta V_\beta$ is in the original system of neighborhoods. Then the set of all $\lambda_\beta V_\beta$ can be written $\{V_\gamma\}_{\gamma \epsilon I}$, where $I \subseteq J$. It is easy to show that I is a directed set under the ordering which it inherits from J.

DEFINITION 4.1. Let the measure $\mu(b)$ of an element $b \epsilon B$ be defined by the equation $\mu(b) = \lim_{\delta \epsilon I} p_\delta(b)$.

If $\alpha \geq \beta$, then $p_\alpha(b) \geq p_\beta(b)$; hence the quantity $\mu(b)$ must exist for all $b \epsilon B$ (possibly as $+\infty$). Clearly, μ is finitely additive, nonnegative, and different from zero except on the zero element of B.

LEMMA 4.2. *The measure* $\mu(b)$ *is countably additive.*

Proof. Let $b = \Sigma_{k=1}^\infty b^k$, where $b^k \cap b^i = 0$ if $k \neq i$. If $\mu(b) = \infty$, then, for any number M, there exists a γ such that $p_\gamma(b) > M$. Then

$$\sum_{k=1}^{\infty} \mu(b^k) \geq \sum_{k=1}^{\infty} p_\gamma(b^k) = p_\gamma(b) > M.$$

Since M is arbitrary, $\Sigma_{k=1}^\infty \mu(b^k) = \infty$. On the other hand, if $\mu(b) < \infty$, then, for any $\varepsilon > 0$, there exists a γ such that $p_\gamma(b) > \mu(b) - \varepsilon$. Then

$$\sum_{k=1}^{\infty} \mu(b^k) \geq \sum_{k=1}^{\infty} p_\gamma(b^k) = p_\gamma(b) > \mu(b) - \varepsilon.$$

Since ε is arbitrary, we must have $\Sigma_{k=1}^\infty \mu(b^k) \geq \mu(b)$. Finally, for any n, we have $\Sigma_{k=1}^n \mu(b^k) = \mu(\Sigma_{k=1}^n b^k) \leq \mu(b)$. It follows that $\Sigma_{k=1}^\infty \mu(b^k) \leq \mu(b)$.

DEFINITION 4.2. $B' = \{b \epsilon B \mid \mu(b) < \infty\}$,

$$B'' = \{b \epsilon B \mid \mu(b) = p_\delta(b) \text{ for some } \delta\}.$$

Note that $B'' \subseteq B' \subseteq B$. If $\mu(b) = p_\delta(b)$, then b_δ determines b. Indeed, suppose that one of the components b_β of b is known, and that $\mu(b) = \bar{p}_\beta(b_\beta)$. If $\alpha < \beta$, then b_α is determined by the mapping $\pi_{\alpha\beta}$. If $\alpha > \beta$, then there is a unique $b_\alpha \epsilon \pi_{\beta\alpha}^{-1}(b_\beta)$ such that $\bar{p}_\alpha(b_\alpha) = \bar{p}_\beta(b_\beta) = \mu(b)$. This follows from the representations of E_α and E_β. Therefore all components of b are known, and b is determined.

LEMMA 4.3. *Every* $b \epsilon B$ *is a union of elements of* B''.

Proof. Let $b = \{\cdots, b_\alpha, \cdots\}$. From above, an element $b^\alpha \epsilon B''$ can be constructed so that its αth component is b_α, and so that $\mu(b^\alpha) = \bar{p}_\alpha(b_\alpha)$. If this is done for all components of b, then $b^\alpha \leq b$ for all α, and $\bigcup_\alpha b^\alpha = b$.

Apply the generalized Stone representation theorem to B"; then B" is isomorphic to $\mathfrak{B}''$, the ring of all compact-open subsets of a totally disconnected, locally compact Hausdorff space S. In such a space, every open set is the union of compact open sets. By virtue of Lemma 4.3, the correspondence between B" and $\mathfrak{B}''$ can be extended to a correspondence between B and the set $\mathfrak{B}$ of all open sets in S. This extended mapping preserves the lattice operation of union. Note that $\mathfrak{B}$ is not a Boolean ring.

Let $\sigma(\mathfrak{B}'')$ be the smallest σ-ring in S which contains $\mathfrak{B}''$. Extend the measure μ from $\mathfrak{B}''$ to $\sigma(\mathfrak{B}'')$, by the standard method; the measurable subsets of S are the elements of $\mathfrak{B}$, modulo sets of measure zero; measurable sets not in $\sigma(\mathfrak{B}'')$ have measure $+\infty$.

THEOREM 4.1. *Every complete, locally* 1-*convex vector lattice which satisfies condition* (V) *is lattice-isomorphic and topologically homeomorphic to a space* $\mathfrak{L}^1(S)$.

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