A THEOREM OF E. HOPF

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In 1948, E. Hopf published [2] a remarkable theorem to the effect that *the total curvature of a closed surface without conjugate points is nonpositive and vanishes only if the surface is flat.* (Here a Riemannian manifold is said to be without conjugate points if no geodesic contains a pair of mutually conjugate points.) Thanks to the Gauss-Bonnet formula, the latter part of this theorem may be paraphrased: a torus without conjugate points is flat. We have been able to modify Hopf's proof to obtain the following result.

THEOREM. *The integral of the scalar curvature (contracted Riemann tensor) of a compact C^4 Riemannian manifold without conjugate points is nonpositive, and it vanishes only if the metric is locally euclidean.*

Here, however, the Gauss-Bonnet-Allendoerfer-Chern-Weil-Fenchel formula does not apply, so that whether an n-dimensional torus without conjugate points is flat is still an open question.

1. ORDINARY DIFFERENTIAL EQUATIONS

Consider the real $m \times m$ matrix differential equation in one independent variable,

$$A''(s) + K(s)A(s) = 0, \qquad \text{(J)}$$

where $K(s)$ is continuous in s and symmetric. (All differentiations, denoted by dashes, and integrations are entry-wise.) Assume that the solution $A(s)$ with $A(0) = 0$ and $A'(0) = I$ (identity matrix) is such that $\det A(s) \neq 0$ for $s \neq 0$. (This corresponds to the nonconjugacy hypothesis.) Then most of the formalism of the one dependent variable case carries over; in particular, the Wronskian of two solutions A and B, $(A')^*B - A^*B'$, is constant (* denotes transpose). Putting $A = B$, we find that $A'A^{-1}$ is symmetric for $s \neq 0$. Setting

$$B_c(s) = A(s)\int_s^c A^{-1}(t)[A^{-1}(t)]^*\,dt,$$

we see that B_c is a solution of (J) for $0 < s < c$ such that

$$B_c(0) = \lim_{s \to 0+} B_c(s) = I$$

and $B_c(c) = 0$. Since the integrand is symmetric and positive definite, and

$$B_c(s) - B_d(s) = A(s)[B_c'(0) = B_d'(0)],$$

Received January 17, 1957.

This research, presented to the American Mathematical Society conference on Differential Geometry in the Large, was partially supported by the Office of Naval Research, contract N6ori-105.

the term in brackets is symmetric and positive definite if $0 < d < c$. (We shall use the same symbol B_c to denote the solution of (J) defined for all s and equal to the integral expression when the latter exists.) Set

$$B_{-1}(s) = A(s)N_c + B_c(s),$$

where $N_c = -A^{-1}(-1)B_c(-1)$. Another argument with the Wronskian shows that N_c is symmetric. $B_{-1}(s)$ is a solution of (J) with $B_{-1}(-1) = 0$, $B_{-1}(0) = I$, and

$$B'_{-1}(0) - B'_c(0) = N_c.$$

Now N_c is positive definite for every positive c. It is sufficient to show this for N_c^{-1}, and differentiation at $t = 0$ reveals that $B_{-1}^{-1}(t)A(t)$ is positive definite for small positive t, consequently for all positive t (in particular, for $t = c$), since its determinant is positive for positive t. Hence the set of positive definite matrices $\{B'_c(0) - B'_1(0) \mid c > 1\}$ is monotone increasing in c and bounded above by

$$B'_{-1}(0) - B'_1(0).$$

The existence of the least upper bound for this set is clear, and we obtain the first part of

LEMMA 1. a) $\lim_{c \to \infty} [B'_c(0) - B'_1(0)] = Q$ *exists and is symmetric.*

b) $\lim_{c \to \infty} B_c(s) = D(s)$ *exists (uniformly for bounded* s *intervals).* D(s) *is a solution of* (J) *such that* $D(0) = I$, $D'(0) = Q + B'_1(0)$, *and* $\det D(s) \neq 0$ *for all* s.

Part (b) of the lemma is a consequence of the continuous dependence of solutions of equation (J) on the initial data.

A computation now shows that $U(s) = D'(s)D^{-1}(s)$ is a symmetric solution, defined for all s, of the Riccati matrix equation

$$\text{(R)} \qquad U'(s) + U^2(s) + K(s) = 0, \qquad -\infty < s < \infty.$$

Moreover, the construction of U(s) is independent of the position of $s = 0$, in the following sense:

LEMMA 2. *If* $Z(s; a) = \lim_{b \to \infty} Z(s; a, b)$, *where* $Z(s; a, b)$ *is the solution of* (J) *with* $Z(a; a, b) = I$ *and* $Z(b; a, b) = 0$, *then* $Z'(s; a)Z^{-1}(s; a) = U(s)$.

The proof of Lemma 2 is the same as the corresponding result in [2], and it will therefore be omitted.

In a system of differential equations such as (R), it is often possible to apply standard Sturm comparison techniques to the inner product $(U(s)x, x)$ for constant vectors x.

LEMMA 3. *If* $(K(s)x, x) \geq -R^2$ *for every unit vector* x *and all* s, *then* $|(U(s)x, x)| \leq R$ *for all* s, *and consequently* U(s) *is uniformly bounded.*

Proof. Suppose $(U(t_0)x, x) > r > R$ for some t_0 and unit x. There is a number d such that

$$r \coth(rt_0 - d) = (U(t_0)x, x);$$

set $V(t) = [r \coth(rt - d)]I$. Then V is a solution of the equation $V' + V^2 - r^2I = 0$, for $t \neq d/r$. Put $f(t) = ([U(t) - V(t)]x, x)$. Then

$$f'(t_0) + (U^2(t_0)x, x) - (V^2(t_0)x, x) + (K(t_0)x, x) + r^2 = 0.$$

But

$$\text{(S)} \qquad (V^2(t_0)x, x) = r^2 \coth^2(rt_0 - d) = (U(t_0)x, x)^2 \leq (U(t_0)x, U(t_0)x) = (U^2(t_0)x, x),$$

by Schwarz's inequality and the symmetry of U. Therefore $f'(t_0) < 0$, and hence $f(t) < 0$ for $t > t_0$. The remainder of the proof follows Lemma 2.1 of [1]; the only additional information needed is inequality (S).

In addition, if $K(s, P)$ depends measurably on the (measure-space) variable P, then $U(s, P)$ is also measurable. (This is proved exactly as in [2].)

2. APPLICATION TO GEOMETRY

Let M be an n-dimensional compact C^4 Riemannian manifold with no conjugate points, B its bundle of orthonormal frames, and T the unit tangent bundle. Let the natural projection of B onto T be given by $(x; e_1, \cdots, e_n) \to (x, e_n)$. The geodesic flow of M is defined to be the one-parameter group of homeomorphisms of T obtained by sending the element $P = (x, e_n)$ after time t into the unit tangent vector P_t at the end of the (directed) geodesic segment of length t with initial conditions (x, e_n). This flow is measure-preserving when one uses the natural volume element $dm = dV\,do$, where dV is the volume element on M and do is the measure on the unit $(n - 1)$-sphere.

By fixing an element $(x; e_1, \cdots, e_n)$ of B, a set of Fermi coordinates is specified along the geodesic on M with initial element (x, e_n). In these coordinates, with s as arc-length, the Jacobi equations become

$$\frac{d^2}{ds^2} y^i(s) + K^i_j(P_s)\, y^j(s) = 0,$$

where the indices run from 1 to $n - 1$, and where $K^i_j(P_s)$ is the curvature tensor contracted in the direction P_s ($P_0 = (x, e_n)$). The hypothesis that there be no conjugate points enables us to apply the results of Section 1, and to obtain a well-defined symmetric matrix $U(s; x, e_1, \cdots, e_n)$ which is a solution for all s of the equation (R), measurable in the bundle variables. If O is an $(n - 1)\times(n - 1)$ orthogonal matrix which accomplishes a change of frame (leaving e_n fixed), the equation becomes

$$O\,U'(s)\,O^{-1} + O\,U^2(s)\,O^{-1} + O\,K(s)\,O^{-1} = 0.$$

Therefore (tr denotes trace)

$$\operatorname{tr} U' + \operatorname{tr} U^2 + \operatorname{tr} K = 0$$

is an equation in functions of $(s, P) = P_s$ only. By Lemma 2, $\operatorname{tr} U$ and $\operatorname{tr} U^2$ are well-defined functions of P_s, regardless of the choice of initial element for the geodesic. Integrating with respect to s, we get

$$\operatorname{tr} U(P_1) - \operatorname{tr} U(P) + \int_0^1 \operatorname{tr} U^2(P_s)\, ds + \int_0^1 K_i^i(P_s)\, ds = 0 .$$

Now integrate with respect to dm over all of T, and use the fact that dm is invariant with respect to the geodesic flow. We find that

$$0 = \int_T \int_0^1 \operatorname{tr} U^2(P_s)\, ds\, dm + \int_T \int_0^1 K_i^i(P_s)\, ds\, dm = \int_T \operatorname{tr} U^2(P)\, dm + \int_T K_i^i(P)\, dm .$$

In terms of local coordinates, with $P = (x, e_n)$ and e_n having components v^j, $K_i^i(P) = K_{jik}^i(x)\, v^i v^k$; hence we may evaluate the last integral as follows:

$$\int_T K_i^i(P)\, dm = \int_M \int_{S^{n-1}} K_{jik}^i(x)\, v^j v^k\, do\, dV = \frac{\omega_{n-1}}{n} \int_M K(x)\, dV ,$$

where $K(x)$ is the scalar curvature of M (the contracted Ricci tensor). The final formula from which the theorem follows is

$$\int_T \operatorname{tr} U^2(P)\, dm = -\frac{\omega_{n-1}}{n} \int_M K(x)\, dV .$$

Now, because U is symmetric, $\operatorname{tr} U^2$ equals the sum of the squares of all components of U. But if U vanishes identically, so must the curvature tensor, since P is arbitrary.

REFERENCES

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