MONOTONE MAPPINGS OF MANIFOLDS, II

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1. INTRODUCTION

In [5], I showed that if S is an orientable, n-dimensional generalized closed manifold (that is, an n-gcm), and f(S) = S' is an (n - 1)-monotone mapping of S onto an at most n-dimensional nondegenerate Hausdorff space S', then S' is an orientable n-gcm of the same homology type as S. In the algebraic apparatus used in defining concepts such as homology and monotoneity, a fixed field was assumed as coefficient group. However, if monotoneity is defined over the integers (see also [2]), then S' need only be assumed to be finite-dimensional.

It is the purpose of the present paper to treat the noncompact and nonorientable cases. Two new conditions enter here. In the first place, for the noncompact case the mapping f must be assumed to be *proper*; that is, the counter-images $f^{-1}(M)$ of compact sets M must be compact. The necessity for this is seen from the simple example where $S = E^1$, the open real number interval $[x \mid 0 < x < 1]$. Let S' be the subspace of the cartesian plane consisting of the sets

$$A = [(x, y) \mid 0 < x \leq 1/2,\ y = 0], \qquad B = [(x, y) \mid 4x^2 - 4x + 4y^2 - 4y + 1 = 0],$$

and let p = (1/2, 0). Let f(S) = S' be the identity on A, and map the open interval $1/2 < x < 1$ onto B - p homeomorphically. Then f is monotone but not proper, and S' is not a generalized manifold.

In the second place, for the nonorientable case, it must be assumed, for each $x' \in S'$, that there exists in S an orientable submanifold (more precisely, an orientable n-dimensional generalized manifold, that is, an orientable n-gm) containing $f^{-1}(x')$. This is shown, for the case n = 3, by the following example (the assumption is apparently unnecessary if n = 2; for the classical case, see [3; Lemma 2]): Let P^2 be the projective plane, and S^1 the 1-sphere, and let $S = P^2 \times S^1$; let the field of coefficients be the integers mod 3. Then S is a nonorientable 3-manifold. For some $x \in S^1$, let $M = P^2 \times x$. The open sets containing M that are topologically similar to $P^2 \times E^1$ form a complete system of neighborhoods for M; however, they are nonorientable 3-manifolds. Hence M, although it is acyclic, has no neighborhoods that are orientable 3-manifolds (since the existence of such would induce orientations on sufficiently small elements of the complete neighborhood system described above). Note that there exists a mapping f(S) = S', where S' is the quotient space of S' in which the only points of S that coalesce are the points of M, which is 2-monotone, and that S' is not a 3-gm.

2. THE ORIENTABLE CASE

In a previous paper [6], I have shown that if S and S' are locally compact spaces, and f(S) = S' is a proper, n-monotone (n > 0), continuous mapping, then $\mathfrak{H}^n(S) = \mathfrak{H}^n(S')$. A similar argument shows that f induces a homomorphism of

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$\mathfrak{H}^{n+1}(S)$ onto $\mathfrak{H}^{n+1}(S')$, for $n \geq 0$. We recall that by n-*monotone* is meant that counter-images of points are r-acyclic for all $r \leq n$, provided the coefficient domain is an algebraic field $\mathfrak{F}$, as we shall generally assume throughout; for other coefficient groups, the term is used to denote *Vietoris mapping of order* n as defined by Begle [1; p. 537]. The group $\mathfrak{H}^n(S)$ is the *infinite homology group;* it is isomorphic with the group $H^n(\hat{S}; \hat{p})$, where $(\hat{S}; \hat{p})$ is the compact pair formed by the one-point compactification $\hat{S}$ of S by addition of an ideal point $\hat{p}$.

For the purposes of the present paper, we need an extension of the Vietoris-Begle theorem (see [1], [2], [4]) to relative homology. This is embodied in the following theorem.

THEOREM 1. *Let* $f(S) = S'$ *be a proper,* n-*monotone* $(n > 0)$, *continuous mapping, where* S *and* S' *are locally compact spaces, and let* A', B' *be closed subsets of* S' *such that* $A' \supset B'$. *Then* $\mathfrak{H}^n(S{:}\ S, B; S, A) = \mathfrak{H}^n(S'{:}\ S', B'; S', A')$, *where* $A = f^{-1}(A')$, $B = f^{-1}(B')$. *Also,* f_* *induces a homomorphism of* $\mathfrak{H}^{n+1}(S{:}\ S, B; S, A)$ *onto* $\mathfrak{H}^{n+1}(S'{:}\ S', B'; S', A')$.

(By $\mathfrak{H}^n(S{:}\ S, B; S, A)$ we denote the n-dimensional homology group formed by the group of n-cycles of S mod B, modulo its subgroup of cycles that bound mod A; see [8; p. 166, Definitions 18, 28]. Theorem 1 holds for n-monotoneity over any group G).

Proof of Theorem 1. Let $U' = S' - B'$, $V' = S' - A'$, $U = f^{-1}(U')$, $V = f^{-1}(V')$; note that $U = S - B$, $V = S - A$, and that $U \supset V$. Consider the diagram

$$\begin{array}{ccc} \mathfrak{H}^n(U) & \xrightarrow{\quad\phi\quad} & \mathfrak{H}^n(U') \\ \downarrow j_* & & \downarrow j'_* \\ \mathfrak{H}^n(V) & \xrightarrow[\quad\psi\quad]{} & \mathfrak{H}^n(V') \end{array}$$

where ϕ and ψ are isomorphisms induced by the mappings $f|U$ and $f|V$, respectively, and j_*, j'_* are the homomorphisms induced by the inclusions $j{:}\ V \to U$, $j'{:}\ V' \to U'$, respectively. That ϕ and ψ exist follows from the theorem cited above.

The commutativity relation

(1) $$\psi j_* = j'_* \phi$$

holds. In addition, the groups $\mathfrak{H}^n(S{:}\ S, B; S, A)$ and $j_* \mathfrak{H}^n(U)$ are isomorphic, as are also the corresponding groups $\mathfrak{H}^n(S'{:}\ S', B'; S', A')$ and $j'_* \mathfrak{H}^n(U')$. Applying (1) to $\mathfrak{H}^n(U)$, we get

(2) $$\psi j_* \mathfrak{H}^n(U) = j'_* \mathfrak{H}^n(U')$$

and hence, from (2), the desired relation

$$\psi \mathfrak{H}^n(S{:}\ S, B; S, A) = \mathfrak{H}^n(S'{:}\ S', B'; S', A').$$

For the case of dimension $n + 1$, the proof is similar.

LEMMA 1. *Let* $f(S) = S'$ *be a proper continuous mapping of a locally compact space* S *onto a locally compact space* S'; *for* $x' \in S'$, *let* $M = f^{-1}(x')$, *and let* P *be an open subset of* S *containing* M. *Then there exists an open subset* P' *of* S' *containing* x' *such that* $f^{-1}(P') \subset P$.

Proof. Let V' be an open subset of S' containing x' such that $\overline{V'}$ is compact. Then $V = f^{-1}(V')$ is an open subset of x containing M such that $\overline{V}$ is compact (since f is proper and $\overline{V} \subset f^{-1}(\overline{V'})$). Let Q be an open subset of S containing M such that $\overline{Q} \subset V \cap P$.

Now the set $\overline{V} - Q$ is a closed set, and hence the set $f(\overline{V} - Q) = F'$ is closed ($\overline{V}$ is compact, and $f|\overline{V}$, as a mapping of a compact space, is a closed mapping). And as a closed subset of the compact set $\overline{V'}$, F' is compact, and hence $F = f^{-1}(F')$ is compact and hence closed. Since $M \cap F$ is empty, the set $P_1 = V - F$ is an open set containing M. The set $P' = f(P_1) = V' - F'$ is the desired set.

THEOREM 2. *Let* $f(S) = S'$ *be an* $(n-1)$*-monotone, proper, continuous mapping of an orientable,* n*-dimensional, generalized manifold* S *onto an at most* n*-dimensional, locally compact, nondegenerate Hausdorff space* S'. *Then* S' *is an orientable,* n*-dimensional, generalized manifold of the same homology type as* S.

Before proceeding to the proof of Theorem 2, we recall that an orientable, n-dimensional, generalized manifold, or, briefly, an orientable n-gm, is a locally compact space S such that (1) the dimension of S is n; (2) S is colc^{n-1}; (3) for each $x \in S$, $p_n(S, x) = 1$; (4) if F is a proper closed subset of S, then every infinite n-cycle on F bounds on S (see [8; p. 254]); and (5) the group $\mathfrak{H}^n(S)$ is of positive dimension. (An n-gm is defined in precisely the same way, except that (5) is replaced by the stipulation that S is connected.)

Proof of Theorem 2. Conditions (4) and (5) imply that S is connected, and hence S' is connected.

We note first that f is n-monotone. For consider a set $F = f^{-1}(x)$, where $x \in S'$. Since f is proper, F is compact, and an n-cycle Z^n on F would be a compact cycle such that $Z^n \sim 0 \bmod \hat{p}$, where $\hat{p}$ is a point of compactification for S. But on a cofinal set of n-dimensional coverings, this implies that $Z^n = 0$.

Since f is n-monotone, $\mathfrak{H}^n(S) = \mathfrak{H}^n(S')$ by the theorem cited above, and it follows that $\mathfrak{H}^n(S')$ is of positive dimension.

We prove next that S' satisfies condition (4). If K' is a proper closed subset of S', then $K = f^{-1}(K')$ is a proper closed subset of S. And since $f|K$ is n-monotone, $\mathfrak{H}^n(K) = \mathfrak{H}^n(K')$. Now it follows from conditions (1) and (4) that if F is any proper closed subset of S, then $\mathfrak{H}^n(F) = 0$. We conclude, therefore, that $\mathfrak{H}^n(K') = 0$.

That S' satisfies condition (2) may be shown as follows: Let $x \in S'$, and suppose U' is any open subset of S' containing x. Then $U = f^{-1}(U')$ is an open subset of S containing the set $M = f^{-1}(x)$. With an appropriate choice of U', we may assume that $\overline{U}$ is compact. By [5; Thm. 3], there exists an open set Q such that $M \subset Q \subset \overline{Q} \subset U$ and such that $H^r(S: \overline{Q}, 0; \overline{U}, 0) = 0$ for all $r < n$. By Lemma 1, there exists an open subset V' of S' such that $x \in V' \subset U'$ and $V = f^{-1}(V') \subset Q$. Clearly, $H^r(S: \overline{V}, 0, \overline{U}, 0) = 0$ for all r such that $0 < r < n$. And by [8; p. 258, Thm. VIII 5.9], $H^r(S: S, S - U; S, S - V) = H^{n-r}(S: \overline{V}, 0; \overline{U}, 0) = 0$ for all r such that $0 < r < n$. (Since $\overline{U}$, $\overline{V}$, and so forth are compact, the $\mathfrak{H}$ and H groups are here the same.) By Theorem 1,

$$H^r(S': S', S' - U'; S', S' - V') = H^r(S: S, S - U; S, S - V) = 0,$$

and since

$$H^r(S': S', S' - U'; S', S' - V') = H_r(S': V', 0; U', 0) = 0$$

by [8; p. 166, Thm. V 18.31], it follows that S is r-colc for all r such that $0 < r < n$. Hence S is $colc^{n-1}$ (see [8; p. 192, Cor. 6.12]).

To show that S' satisfies condition (3), let x, U', U, M be as in the preceding paragraph. Let Q be an open connected set such that $M \subset Q \subset \overline{Q} \subset U$; the existence of such a Q follows from the connectedness of M and the local connectedness of S. Let V' be an open subset of S' such that $x \in V' \subset U'$, and such that the set $V = f^{-1}(V')$ is a subset of Q. (That such a set exists follows from Lemma 1). By [8; p. 258, Thm. VIII 5.9], $H^0(S: \overline{V}, 0; \overline{U}, 0) = H^n(S: S, S - U; S, S - V)$. And by Theorem 1,

$$H^n(S': S', S' - U'; S', S - V') = H^n(S, S - U; S, S - V).$$

Since $\dim H^0(S: \overline{V}, 0; \overline{U}, 0) = 1$, it follows that $\dim H^n(S': S', S' - U'; S', S' - V') = 1$. And since V' is unrestricted, except that $f^{-1}(V')$ must be a subset of Q, the number $p^n(x; U')$ is 1 (see [8; p. 190, Section 6.6]). Since U' was also unrestricted (except that $f^{-1}(U') = U$ has compact closure), the number $p^n(S', x')$ is also 1.

THEOREM 2'. (This is the same as Theorem 2, except that S' is only assumed to be *finite-dimensional*, and the mapping f is assumed to be (n - 1)-*monotone over the integers*).

Proof. The proof is similar to that of Main Theorem B of [5], except that it must be observed that Lemma 4 of [5] holds when f is a *proper* n-monotone mapping of a locally compact space S, as does also Begle's proof [2; Section 6, p. 542] that monotoneity defined in terms of acyclicity of inverse images over a field implies monotoneity as defined in Definition 3' of [5].

3. THE NONORIENTABLE CASE

In both [5] and Theorem 2 above, we have assumed orientability of the manifold, throughout. We now consider what can be said when this assumption is not made.

LEMMA 2. *If* $f(S) = S'$ *is a proper mapping of a locally compact space* S *onto locally compact space* S', *then* f *is a closed mapping.*

Proof. Let F be a closed subset of S, and let $F' = f(F)$. Suppose there exists a point x' of S' - F' such that x' is a limit point of F'. Let $M = f^{-1}(x')$; then M is compact, since f is proper, and consequently M is a closed subset of S - F.

There exists, by [8; p. 101, Lemma IV 1.9], an open subset U of S such that $M \subset U \subset \overline{U} \subset S - F$. By Lemma 1, there exists an open set P' of S', containing x', such that the set $P = f^{-1}(P')$ is a subset of U. But this is impossible, since $P' \cap F'$ is not empty.

THEOREM 3. *Let* $f(S) = S'$ *be an* (n - 1)-*monotone, proper, continuous mapping of an* n-gm S *onto an at most* n-*dimensional, nondegenerate locally compact Hausdorff space* S', *such that for each* $x' \in S'$, *the set* $f^{-1}(x')$ *lies in an orientable* n-gm *in* S. *Then* S' *is a locally orientable* n-gm *of the same homology type as* S.

Proof. In the first place, S' is an n-gm of the same homology type as S. To show this, we may proceed as in the proof of Theorem 2. However, in proving that S' satisfies conditions (2) and (3), use is made of Theorem VIII 5.9 of [8; p. 258], which is a generalized form of Poincaré duality and requires orientability of the manifold in question. In order to justify this in the present proof, we may use the fact that if M is a set $f^{-1}(x')$ $(x' \in S')$, then M is contained in an open subset U of S such that U is an orientable n-gm. (Every n-gm which is a subset of a locally orientable n-gm S is an open subset of S. This is proved in [7].) By Lemma 1, the set U of the proof of Theorem 2 may be so selected that its inverse $f^{-1}(U)$ lies in the orientable manifold U whose existence is asserted above. The argument used in Theorem 2 now follows if we make use of the excision property of homology theory (that S' is at least n-dimensional follows from the fact that $p_n(S', x') > 0$ for all $x' \in S'$).

We have, then, only to show that S' is locally orientable; in other words, that every point of S' has a neighborhood which is an orientable n-gm. Let $x' \in S'$ and $M = f^{-1}(x')$. By hypothesis, there exists an open set P in S containing M such that P is an orientable n-gm. By Lemma 1, there exists an open set U' in S' containing x' and such that $U = f^{-1}(U')$ is a subset of P. The component V of U containing M is open (since S is locally connected) and constitutes an orientable n-gm. (This is proved in [7; Thm. 2.2].) And since f is 0-monotone, $f^{-1}f(V) = V$. Thus the mapping $f_1: V \to f(V)$, where $f_1 = f|V$, is an (n - 1)-monotone mapping. Consequently f(V) is an orientable n-gm, by Theorem 2.

To show that f(V) is open, we need only note that S - V is closed; whence, by Lemma 2, f(S - V) = S' - f(V) is closed.

THEOREM 3'. (This is the same as Theorem 3, except that S' is only assumed to be *finite-dimensional*, and the mapping f is assumed to be (n - 1)-*monotone over the integers*)

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