ON CURVATURE AND CHARACTERISTIC OF HOMOGENEOUS SPACES

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1. This note is concerned with two topics: We show, with a simple geometrical proof, that the Riemannian curvature of a homogeneous space is nonnegative (Thm. I); this had been proved earlier by E. Cartan for symmetric spaces [3]. And we give a new proof for the fact that the Euler-Poincaré characteristic of such a space is nonnegative (Thm. II). It is apparently still unknown whether Theorem II can be deduced from Theorem I by way of the generalized Gauss-Bonnet formula.

Let G be a compact connected Lie group, U a closed subgroup of G, and M = G/U the homogeneous space formed by the left cosets of U; let p be the natural projection of G onto M. The space M is a manifold, and it carries a differentiable structure of class C^∞, induced by p (the C^∞-functions on M are identified with the C^∞-functions on G that are constant on the cosets of U). The group G admits Riemannian metrics that are invariant under left and right translations (bi-invariant); let such a metric be chosen. There is then a Riemannian metric in M, induced in a natural way (see below for a description); and this metric is invariant under the customary action of G on M.

THEOREM I. *All values of the sectional Riemannian curvature of* M, *in the induced Riemannian metric, are nonnegative.*

For a 2-section Σ (two-dimensional subspace of the tangent space) at a point x we denote the sectional Riemannian curvature in direction Σ by $K(x, \Sigma)$.

2. We consider first the special case where M is itself a group.

PROPOSITION 2.1. *All values of the sectional Riemannian curvature in a Lie group* G, *under a bi-invariant metric, are nonnegative. The sectional curvature in direction* Σ *at the identity* e *vanishes if and only if* Σ *generates an abelian subgroup, that is, if and only if the one-parameter groups generated by the vectors in* Σ *commute.*

Proof. It is well known that the geodesics (parametrized proportionally to arc length) of G are exactly the 1-parameter groups in G and their cosets.

Because of transitivity, it is sufficient to consider 2-sections at e. Let Σ be such a section, let X, Y be two vectors spanning Σ (we may and shall assume that they are orthogonal to each other), and let x(t), y(t) be two one-parameter groups whose tangent vectors for t = 0 are X and Y, respectively. We map the (t_1, t_2)-plane E^2 into the group G by the map ϕ defined by $\phi(t_1, t_2) = x(t_1) \cdot y(t_2)$. This is easily seen to be a regular map; that is, the differential $\dot{\phi}$ is nonsingular throughout. In fact, the image under $\dot{\phi}$ of the horizontal, respectively vertical unit vector at (t_1, t_2) $\left[\text{that is, } \frac{\partial}{\partial t_1}, \text{ respectively } \frac{\partial}{\partial t_2}, \text{ in customary notation}\right]$ is $x(t_1) \cdot X \cdot y(t_2)$, respectively $x(t_1) \cdot Y \cdot y(t_2)$; we have denoted the action of a left or right translation on a vector simply by left or right multiplication. Since translations are isometries, the two image vectors are independent and orthogonal to each other. The ϕ-images of

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the straight lines t_i = constant $(i = 1, 2)$ are cosets of one-parameter groups in G. All this implies that for the Riemannian metric in E^2 induced by ϕ, the straight lines just mentioned are geodesics; the two families of geodesics corresponding to $i = 1$ and $i = 2$ meet at right angles throughout. It is well known from differential geometry that the induced metric in E^2 is flat, that is, has Riemannian curvature identically zero; this follows, for example, from the Gauss-Bonnet formula applied to small squares. We now apply the Lemma of Synge [7, Theorem XIII]:

Let S *be a two-dimensional subspace of a Riemannian manifold* V *that passes through a geodesic* g *of* V. *Then at every point of* g *the Riemannian (Gaussian) curvature of* S *is less than or equal to the sectional Riemannian curvature of* V *in the* 2*-direction tangent to* S. *Equality holds only if the unit normal vector to* g *in* S *has vanishing covariant derivative (relative to* V).

It follows immediately that the curvature of G at (e, Σ) is nonnegative, namely greater than or equal to that of the submanifold $\phi(E^2)$.

To discuss the case where $K(e, \Sigma) = 0$, we note first that parallel transport of the vector Y along $x(t)$ from 0 to t is obtained by forming $x(t/2) \cdot Y \cdot x(t/2)$. This results easily from the well-known remark of E. Cartan's that, inversion being an isometry, the parallel field along $x(t)$ for $t \geq 0$ that is generated by Y goes over into the parallel field along $x(t)$ for $t \leq 0$ generated by $-Y$, if $x \to x^{-1}$ is applied.

Since left and right translations are isometries, $K(e, \Sigma) = 0$ implies the vanishing of the curvature of G for the 2-section tangent to $\phi(E^2)$ at any point $\phi(t_1, t_2)$. By Synge's lemma we have $K(e, \Sigma) = 0$ exactly if the normal field to the curve $t_2 = 0$, which is given by $Y(t_1) = x(t_1) \cdot Y$, is parallel; by the remark just made, this means that $x(t_1/2) \cdot Y \cdot x(t_1/2) = x(t_1) \cdot Y$. Writing t for $t_1/2$, we obtain $x(-t) \cdot Y \cdot x(t) = Y$ for all t; this is of course equivalent to the commuting of $x(t)$ and $y(t)$, and Proposition 2.1 is proved.

3. We consider now the quotient space $M = G/U$. The Riemannian metric in M, used in Theorem I, can be described as follows: Let x be a point in M, and let X be a tangent vector of M at x. Let x' be a point of G with $p(x') = x$, and let X' be a vector of G at x', orthogonal to the coset $x' \cdot U$ (a submanifold of G) at x', with $\dot{p}(X') = X$; note that $\dot{p}$ maps the tangent space to G at x' onto the tangent space to M at x, and that the kernel is the tangent space to $x' \cdot U$ at x'. Then the norm $|X|$ of X is defined to be equal to the norm $|X'|$ of X (this is independent of the choice of x' which is involved). The definition implies that $\dot{p}$ maps the normal space of $x' \cdot U$ at x' isometrically onto the tangent space of M at x. For an arbitrary vector Y' at x' we have $|\dot{p}(Y')| \leq |Y'|$, with equality only if Y' is normal to $x' \cdot U$. We call a curve in G transversal if at each of its points its tangent vector is orthogonal to the coset of U through the point. All curves are assumed to be piecewise C^∞.

LEMMA 3.1. *If* $\mathfrak{C}'$ *is a curve in* G, *and* $\mathfrak{C}$ *its projection under* p, *then the inequality* $L(\mathfrak{C}') \geq L(\mathfrak{C})$ *holds between the lengths of* $\mathfrak{C}'$ *and* $\mathfrak{C}$; *equality holds if and only if* $\mathfrak{C}'$ *is transversal.*

The lemma follows immediately from the facts above, since length is the integral of the norm of the tangent vector.

By standard theorems on differential equations, there exists for each curve $\mathfrak{C}$ in M a transversal curve $\mathfrak{C}^*$ in G whose p-projection is exactly $\mathfrak{C}$; moreover, $\mathfrak{C}^*$ is unique up to right translation by elements of U. With the help of Lemma 3.1 and the local minimum properties of geodesics, one concludes that $\mathfrak{C}^*$ is a (transversal) geodesic of G if $\mathfrak{C}$ is a geodesic of M. In other words, the geodesics of M are the images, under p, of the transversal geodesics of G.

4. Let now x be a point of M, and Σ a 2-direction through x. Let x' be a point of G with p(x') = x, and let Σ' be a 2-direction of G at x', orthogonal to $x' \cdot U$ and with $\dot{p}(\Sigma') = \Sigma$. We construct the usual geodesic surface P tangent to Σ; that is, we consider the exponential map ψ of Σ into M, which to each vector X in Σ assigns, on the geodesic that starts at x in direction X, the point corresponding to parameter value (arc length) $|X|$. It is standard that the sectional curvature $K(x, \Sigma)$ equals the Gaussian curvature of P at x. It is also well known that this value can be obtained as follows [1, p. 153]: Let C_r denote the circle of radius r around x in Σ; denote by L_r the length of the curve corresponding to C_r in P or in M, that is, the length of the curve $\psi(C_r)$ [or the length of C_r in the metric induced in Σ under ψ], for small r. Then

4.1 $$K(x, \Sigma) = \lim_{r \to 0} \frac{2\pi r - L_r}{r^3} \cdot \frac{3}{\pi}.$$

Performing the same construction for Σ', we obtain the geodesic surface P', the map ψ' of Σ' into G, the circle C'_r, the curve $\psi'(C'_r)$ of length L'_r, and the relation

4.1' $$K(x', \Sigma') = \lim_{r \to 0} \frac{2\pi r - L'_r}{r^3} \cdot \frac{3}{\pi}.$$

From the connection between the geodesics in M and the transversal geodesics in G we conclude that the image of $\psi'(C'_r)$ under p is $\psi(C_r)$. Lemma 3.1 gives the inequality $L_r \leq L'_r$, for all small r. From 4.1 and 4.1' we conclude that

$$K(x, \Sigma) \geq K(x', \Sigma').$$

Since $K(x', \Sigma') \geq 0$ by Proposition 2.1, we finally have $K(x, \Sigma) \geq 0$, and Theorem I is proved.

To discuss the case $K(x, \Sigma) = 0$, we make the following remark: Let H be a Lie subgroup of G whose Lie algebra is orthogonal to the Lie algebra of U. Then the tangent space to H at any point y of H is orthogonal to the tangent space of the coset $y \cdot U$, by left invariance. It follows that the projection p is an isometry of H into M (not necessarily one-to-one). From this remark it follows that, in the notation above, $K(x, \Sigma)$ vanishes exactly if the left translate of Σ' by $(x')^{-1}$ generates an abelian subgroup. In particular, the sectional curvature of M is positive throughout if and only if there are no 2-dimensional abelian subgroups orthogonal to U. If M is a symmetric space in the sense of Cartan, this condition means that all geodesics are closed (and of the same length); the space is said to be of rank one. It is well known that the spaces of rank one are the spheres, the real, complex or quaternion projective spaces, and the Cayley projective plane.

5. We come now to our second topic.

THEOREM II: *The Euler-Poincaré characteristic* $\chi(M)$ *of* M = G/U *is nonnegative; it is positive exactly if* U *is of maximal rank.*

We recall that the rank of a compact Lie group is the dimension of any maximal torus subgroup. We may assume that U is connected: If necessary, it can be replaced by the e-component; this amounts to going to a covering space of M, and the latter process multiplies the characteristic by a positive integer.

In [5] and in [8], Theorem II is proved with the help of the Lefschetz fixed-point theorem. We begin the proposed new proof with a proposition about arbitrary (continuous) representations. For a vector space V over the reals, we denote by $\Lambda^i(V)$ the exterior powers of V (the skew-symmetric tensors with i indices; see [2]), and by $\Lambda(V)$ the direct sum $\Sigma_1^n \Lambda^i(V)$, with $n = \dim V$.

A linear transformation T of V into itself induces a linear transformation $T^{(i)}$ of each $\Lambda^i(V)$ into itself; $T^{(i)}$ is the pth exterior power of T. With the identity map I and a variable t we form the polynomial $\det(t \cdot T + I)$. The coefficient of t^i in the expansion $\Sigma_0^n c_i t^i$ of this polynomial is then equal to the trace of the transformation $T^{(i)}$. This becomes clear if T is represented by a matrix; the principal minors of T, whose sum is c_i, are the diagonal elements for the matrix representing $T^{(i)}$.

Definition 5.1. Let D be a representation of the compact connected Lie group U by linear transformations of the real vector space V; then D induces a representation $D^{(i)}$ in $\Lambda^{(i)}(V)$; denote by λ_i the "number" of invariant skew i-tensors, that is, the dimension of the subspace I_i of $\Lambda^i(V)$ consisting of those elements v for which $D^{(i)}(x)(v) = v$ for all $x \in U$. The polynomial $\Sigma \lambda_i t^i$ is called the *Poincaré polynomial* $P_D(t)$ of D. The alternating sum $\Sigma(-1)^i \lambda_i$ is called the *(Euler-Poincaré) characteristic* $\chi(D)$ *of* D. We have $\chi(D) = P_D(-1)$.

PROPOSITION 5.2. *The characteristic* $\chi(D)$ *of any continuous representation of* U *is nonnegative;* $\chi(D)$ *is positive exactly if* D *is "of maximal rank," that is, if for each maximal torus group* T *in* U *the zero-vector is the only vector in* V *which is fixed by all* D(x) $(x \in T)$.

6. *Proof of* 5.2. In U we have the usual bi-invariant measure, of total measure 1. It is well known from representation theory that the number of invariants of a representation is equal to the integral over U of the trace of the representation. It follows that $P_D(t) = \int_U \det(t \cdot D(x) + I)$ and, in particular,

6.1 $$\chi(D) = \int_U \det(I - D(x)).$$

As usual, we may assume that D leaves some inner product in V invariant, so that all D(x) are orthogonal transformations. First consider the case where n is odd. Since U is connected, all D(x) preserve orientation. It is well known that for any orientation-preserving orthogonal transformation T in an odd-dimensional space, the relation $\det(T - I) = 0$ holds; 1 is an eigenvalue, as one can see, for example, from the usual normal form for T, or from the identities

$$\det(T - I) = \det(T - T \cdot T') = \det T \cdot \det(I - T') = \det(I - T) = -\det(T - I).$$

It follows that the integrand in 6.1 vanishes identically, and that $\chi(D) = 0$.

Suppose next that n is even. For any orientation-preserving orthogonal transformation T in an even-dimensional space, the inequality

$$\det(I - T) = \det(T - I) \geq 0$$

holds. Again one can see this from the normal form, where it reduces to an inequality of the form

$$\begin{vmatrix} \cos\phi - 1 & -\sin\phi \\ \sin\phi & \cos\phi - 1 \end{vmatrix} \geq 0.$$

Another proof consists in noting that $\det(T - I)$ has the same sign as $\det(t \cdot T - I)$, for large t. (The proofs of Theorem II in [5] and [8] also make use of this fact). Since all $D(x)$ are orientation-preserving, the integrand in 6.1 is now nonnegative, and it follows that $\chi(D) \geq 0$.

We come to the "maximal rank" part of Proposition 5.2. Suppose $\chi(D) = 0$. Then the integrand of 6.1 must vanish identically, since it is never negative. In particular, if x_0 is a generating element of the maximal torus T, then $\det(D(x_0) - I) = 0$, $D(x_0)$ has 1 as eigenvalue, and has therefore a nontrivial fixed vector. This vector is then automatically fixed for all $D(x)$ with $x \in T$.

Conversely, suppose there is a nontrivial fixed vector for the maximal torus T, so that $\det(D(x) - I) = 0$ for $x \in T$. Since every element of U lies on some maximal torus (see [4]) and all maximal tori are conjugate (see [6] for a simple proof), we see that $\det(D(x) - I)$ is identically zero on U, and 6.1 shows $\chi(D) = 0$.

Incidentally, since $T^{(i)}$ and $T^{(n-i)}$ can be considered as adjoints of each other, so that they have the same trace, we have $\lambda_i = \lambda_{n-i}$.

7. We come now to the homogeneous space $M = G/U$. By the de Rham theory, the ith Betti number of M, the rank of the ith cohomology group $H^i(M)$ over the reals (a vector space), is obtainable from differential forms as the rank of the quotient space of the space of closed i-forms (i-cocycles) by the space of derivatives of (i - 1)-forms (i-coboundaries). It is well known that, G being connected, integration over G permits us to restrict consideration to differential forms that are invariant under the action of G on M. Since G is transitive, an invariant differential i-form is determined uniquely by its value at any point of M, for example at the point $e' = p(e)$, the value being an element of the ith exterior power of the adjoint W^* of the tangent space W of M at e'. The differential form determined by an element ω of $\Lambda^i(W^*)$ is invariant exactly if ω is invariant under the transformations which are induced in $\Lambda^i(W^*)$ by the so-called adjoint operation of U on W [any element of U, in its action on M, leaves e' fixed, and determines therefore a linear map of W into itself: the differential of the map at e'], or rather by the adjoint (that is, transposed-inverse) of this representation in W^*. In other words, the graded vector space of invariant differential forms (direct sum over the spaces of forms of various dimensions) can be identified with the graded subspace $I_U = \Sigma_0^n I_i$ of $\Lambda(W^*)$ that consists of the elements invariant under the transformations induced by the adjoint representation of U on W^*. Exterior differentiation, which maps invariant differential forms into invariant differential forms, becomes a linear map d of I_U into itself which raises the degree by 1 (that is, $d(I_i) \subset I_{i+1}$) and satisfies the condition $d \circ d = 0$. The pair (I_U, d) is then a chain-complex K (of vector spaces) of finite type. The cohomology groups $H^i(M)$ of M are thus identified with the homology groups $H^i(K)$ of the complex K. By Proposition 5.2, the characteristic $\chi(K)$ of K, that is, the number $\Sigma (-1)^i \dim I_i$, is nonnegative, since it is just the characteristic of the adjoint representation of U on W^*. It is a classical result that the characteristic of a chain complex (over the reals) of finite type is equal to the characteristic of the graded vector space formed by its homology groups. We have therefore

$$\chi(M) = \sum (-1)^i \dim H^i(M) = \sum (-1)^i \dim H^i(K) = \chi(K) \geq 0,$$

and the inequality of Theorem II is proved.

Again by 5.2, we have $\chi(M) = \chi(K) = 0$ exactly if for each maximal torus T in U there is a nontrivial vector v at e' in M invariant under the adjoint action (since the adjoint action can be assumed orthogonal, the existence of a fixed element in W^*

implies the existence of a fixed element in W). We interpret W as the orthogonal complement of the tangent space of U in the tangent space of G at e; this makes it possible to consider the adjoint action of U on W as induced by inner automorphisms of G by elements of U. Then the existence of v is equivalent to the existence of a 1-parameter group in G, not contained in U, that commutes with T, that is, with the nonmaximality of T. This completes the proof of Theorem II.

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