ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS, II

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Let M^n be a compact differentiable manifold of dimension n, and let

$$x: M^n \to E^{n+N}$$

be a differentiable mapping of M^n into a Euclidean space of dimension $n + N$ with the property that the functional matrix is everywhere of rank n. Then M^n is said to be immersed in E^{n+N}. If x is one-one, it is said to be imbedded in E^{n+N}. To each unit normal vector $\nu(p)$ of an immersed manifold M^n at $p \in M$, we draw through the origin O of E^{n+N} the unit vector parallel to it. This defines a mapping, to be called $\tilde{\nu}$, of the normal sphere bundle B_ν of M^n into the unit hypersphere S_0 about O. In a previous paper [1; this paper will be referred to as TCI], we studied the volume of the image of $\tilde{\nu}$ and called it the *total curvature* of M^n. It will be advantageous to normalize this volume by dividing it by the volume c_{n+N-1} of S_0, c_{n+N-1} being of course an absolute constant. Throughout this paper, we will understand by the total curvature of M^n the normalized one. Then, if $E^{n+N} \subset E^{n+N'}$ $(N < N')$, the total curvature $T(M^n)$ of M^n remains the same, whether M^n is considered as a submanifold of E^{n+N} or of $E^{n+N'}$ (Lemma 1, Section 1). One of the theorems we proved in TCI states that $T(M^n) \geq 2$. We shall show below (Section 1) that the same argument can be used to establish the following more general theorem.

THEOREM 1. *Let* M^n *be a compact differentiable manifold immersed in* E^{n+N}, *and let* β_i $(0 \leq i \leq n)$ *be its* i*th Betti number relative to a coefficient field. Then the total curvature* $T(M^n)$ *of* M^n *satisfies the inequality*

$$T(M^n) \geq \beta(M^n)\,, \tag{1}$$

where $\beta(M^n) = \sum_{i=0}^{n}\beta_i$ *is the sum of the Betti numbers of* M^n.

The right-hand side of (1) depends on the coefficient field. For the real field, the lower bound in (1) cannot always be attained. In fact, we have the following theorem.

THEOREM 2. *If the equality sign holds in* (1) *with the real field as coefficient field, then* M^n *has no torsion.*

For a compact differentiable manifold M^n given abstractly, the total curvature $T(M^n)$ or $T_x(M^n)$ is a function of the immersion $x: M^n \to E^{n+N}$ (N arbitrary). Obviously, the number $q(M^n) = \inf_x T_x(M^n)$ is a global invariant of M^n itself. Theorem 1 says that $q(M^n) \geq \beta(M^n)$. In this connection, there is another invariant $s(M^n)$ of M^n, namely the minimum number of cells in a cell complex covering M^n. Clearly, we have $s(M^n) \geq \beta(M^n)$. If M^2 is a compact orientable surface of genus g, it is easy to see that

$$q(M^2) = s(M^2) = \beta(M^2) = 2 + 2g\,.$$

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Generally, it can be shown that $q(M^n)$ is an integer; but the proof will not be included in this paper. It seems likely that $q(M^n) = s(M^n)$.

Another problem in this order of ideas is the characterization of the immersions of M^n by which the minimal total curvature of M^n is realized, that is, for which $T(M^n) = q(M^n)$. If M^n is homeomorphic to an n-sphere, it is a consequence of Theorem 3 of TCI that such an immersion is characterized by the property that M^n is imbedded as a convex hypersurface in a linear space of dimension $n+1$. The general problem can therefore be regarded as a natural generalization of the theory of convex hypersurfaces in Euclidean space. When M^n is immersed as a hypersurface, that is, when $N = 1$, the Gauss-Kronecker curvature $K(p)$ $(p \in M^n)$, a local invariant of M^n, plays an important role in our problem. It is defined only up to sign when n is odd. The answer to our problem is most complete in the case of compact surfaces imbedded in ordinary Euclidean space $(n = 2, N = 1)$:

THEOREM 3. *A compact orientable surface of genus* g *is imbedded in the three-dimensional Euclidean space with total curvature* $2g + 2$, *if and only if the surface lies at one side of the tangent plane at every point of positive Gaussian curvature.*

For oriented compact hypersurfaces $(N = 1)$ with Gauss-Kronecker curvature $K(p) \geq 0$ for all $p \in M^n$, we have the following theorem.

THEOREM 4. *A compact orientable surface immersed in three-dimensional Euclidean space with Gaussian curvature* ≥ 0 *is imbedded and convex. There are examples of nonconvex compact orientable hypersurfaces, of dimension* ≥ 3, *whose Gauss-Kronecker curvature is everywhere* ≥ 0.

The main point of this theorem is that $K(p)$ is assumed merely to be ≥ 0, and not strictly > 0. In the latter case, a well-known argument due to Hadamard shows that M^n is imbedded as a convex hypersurface. Theorem 4 implies that a conjecture made by us in TCI (p. 318) is true for $n = 2$ and false for $n \geq 3$.

1. TOTAL CURVATURE AND THE SUM OF BETTI NUMBERS

LEMMA 1. *Let* $x\colon M^n \to E^{n+N}$ *be an immersion of a compact differentiable manifold of dimension* n *in* E^{n+N}, *given by*

$$x\colon p \to x(p) = (x^1(p), \cdots, x^{n+N}(p)) \qquad (p \in M^n).$$

Let $x'\colon M^n \to E^{n+N'}$ $(N < N')$, *be the immersion defined by*

$$x'(p) = (x^1(p), \cdots, x^{n+N}(p), 0, \cdots, 0).$$

Then the immersed mainfolds $x(M^n)$ *and* $x'(M^n)$ *have the same total curvature.*

The lemma is intuitively obvious. For if B'_ν is the normal sphere bundle, and $\tilde{\nu}'\colon B' \to S^{n+N'-1}$ the corresponding normal map of the immerson x', then clearly $\tilde{\nu}'$ is the $(N' - N)$-fold suspension of $\tilde{\nu}$ on each fiber. Since $S^{n+N'-1}$ is the $(N' - N)$-fold suspension of S^{n+N-1}, it follows that the ratio of the area covered by $\tilde{\nu}'$ on $S^{n+N'-1}$ to the area covered by $\tilde{\nu}$ on S^{n+N-1} is the same as the ratio of the areas of $S^{n+N'-1}$ and S^{n+N-1}. In spite of this short argument, we give a more analytical proof as follows:

It suffices to prove the lemma for the case $N' - N = 1$. The general case will then follow by induction on the difference $N' - N$.

We follow the notation of TCI, and consider the bundle B of all frames

$$x(p)e_1 \cdots e_{n+N} \qquad (p \in M^n)$$

such that $e_1, \cdots, e_n$ are tangent vectors and $e_{n+1}, \cdots, e_{n+N}$ are normal vectors at $x(p)$. If we put

$$\omega_{n+N,A} = de_{n+N} \cdot e_A \qquad (1 \leq A \leq n+N),$$

then the total curvature is, according to our definition,

$$T_x(M^n) = \frac{1}{c_{n+N-1}} \int_{B_\nu} \omega_{n+N,1} \wedge \cdots \wedge \omega_{n+N,n+N-1} \;,$$

where the integral is taken in the measure-theoretic sense. It is to be pointed out that, as stated in the Introduction, we have inserted the factor $1/c_{n+N-1}$ to normalize the total curvature.

Let a be one of the two unit vectors perpendicular to E in E^{n+N-1}. A unit normal vector at $x'(p)$ can be written uniquely in the form

$$e'_{n+N+1} = (\cos\theta)\, e_{n+N} + (\sin\theta)\, a \qquad \left(-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}\right),$$

where e_{n+N} is the unit vector in the direction of its projection in E^{n+N}. Let

$$e'_{n+N} = (\sin\theta)\, e_{n+N} - (\cos\theta)\, a, \qquad e'_s = e_s \qquad (1 \leq s \leq n+N-1)$$

and

$$\phi_{n+N+1,A} = de'_{n+N+1} \cdot e'_A .$$

Then the total curvature of the immersed manifold $x'(M^n)$ is equal to

$$T_{x'}(M^n) = \frac{1}{c_{n+N}} \int_{B'_\nu} \phi_{n+N+1,1} \wedge \cdots \wedge \phi_{n+N+1,n+N} .$$

Now we have

$$de'_{n+N+1} = (\cos\theta) de_{n+N} + \{-(\sin\theta) e_{n+N} + (\cos\theta) a\} d\theta = (\cos\theta) de_{n+N} - e'_{n+N} d\theta .$$

Since

$$de_{n+N} \cdot e'_{n+N} = -(\cos\theta)(de_{n+N} \cdot a) = (\cos\theta)(e_{n+N} \cdot da) = 0,$$

we find that

$$\phi_{n+N+1,n+N} = de'_{n+N+1} \cdot e'_{n+N} = -d\theta .$$

Also

$$\phi_{n+N+1,s} = (\cos\theta)(de_{n+N} \cdot e_s) = (\cos\theta)\omega_{n+N,s} .$$

It follows that

$$T_{x'}(M^n) = \frac{c_{n+N-1}}{c_{n+N}} \int_{-\pi/2}^{\pi/2} |\cos^{n+N-1}\theta|\, d\theta \cdot T_x(M^n) = \frac{2c_{n+N-1}}{c_{n+N}} \int_0^{\pi/2} \cos^{n+N-1}\theta\, d\theta \cdot T_x(M^n).$$

That $T_{x'}(M^n) = T_x(M^n)$ is then a consequence of the following well-known formulas:

$$c_k = \frac{2\left[\Gamma\left(\frac{1}{2}\right)\right]^{k+1}}{\Gamma\left(\frac{k+1}{2}\right)}, \quad \int_0^{\pi/2} \cos^k\theta\, d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k+2}{2}\right)}.$$

This completes the proof of the lemma.

Since the total curvature is clearly invariant under motions in space, Lemma 1 implies that the total curvature of $x(M^n)$ in E^{n+N} remains unchanged if $x(M^n)$ is considered as a submanifold of a high-dimensional Euclidean space which contains E^{n+N} as a linear subspace.

We wish now to give a proof of Theorem 1. As in the proof of Theorem 1 of TCI, we consider the map $\tilde{\nu}: B_\nu \to S_0^{n+N-1}$ defined by assigning to each unit normal vector the end-point of the unit vector through the origin parallel to it. The total curvature of M^n is by definition the volume of the image of B_ν under $\tilde{\nu}$. The singular points of $\tilde{\nu}$, that is, the points where the functional determinant of $\tilde{\nu}$ is zero, are exactly the points where the quadratic differential form $\nu \cdot d^2x = -d\nu \cdot dx$ is of rank $< n$. By Sard's theorem, their image on S_0^{n+N-1} has measure zero. Hence, for almost all unit vectors ν, the function $\nu \cdot x(p)$ on M^n, with ν fixed, has only nondegenerate critical points. By Morse's inequalities, the total number of critical points is $\geq \Sigma \beta_i(M^n) = \beta(M^n)$. Now the image of B_ν under $\tilde{\nu}$ is the same as the set of points $\nu \in S_0^{n+N-1}$, each counted a number of times equal to the number of critical points of the function $\nu \cdot x(p)$ on M^n. It follows that the measure of the image is $\geq c_{n+N-1}\beta(M^n)$, and hence that the total curvature of M^n is $\geq \beta(M^n)$.

2. IMMERSIONS WITH MINIMAL TOTAL CURVATURE

Proof of Theorem 2.† This theorem follows immediately from Theorem 1. In fact, let $\beta_i(M^n, F)$ be the ith Betti number of M^n with the coefficient field F, and let $\beta(M^n, F)$ be the sum of these Betti numbers. If R denotes the real field and Z_p the field mod p (p a prime), we have

†We are indebted to the referee for this elementary argument. Our original proof makes use of results of R. Thom [3] and of Eilenberg and Shiffman [2, p. 53] concerning the cell decomposition of a manifold on the basis of a real-valued function on it. The result of Eilenberg and Shiffman can be stated as follows: If a compact differentiable manifold M has a differentiable function on it with k nondegenerate critical points of indices $i_1, \cdots i_k$, respectively, then M is of the same homotopy type as a cell complex consisting of k cells of dimensions $i_1, \cdots, i_k$, respectively. Theorem 2 follows immediately from this, because there is a coordinate function with $\beta(M^n, R)$ nondegenerate critical points. The theorem of Eilenberg and Shiffman also gives more information on manifolds satisfying the hypothesis of Theorem 2. For instance, it follows easily that the fundamental group of M^n is isomorphic to its first homology group.

$$\beta_i(M^n, R) \leq \beta_i(M^n, Z_p) \qquad (0 \leq i \leq n).$$

Now by hypothesis, $T(M^n) = \beta(M^n, R)$, and by Theorem 1, $T(M^n) \geq \beta(M^n, Z_p)$, so that $\beta(M^n, R) \geq \beta(M^n, Z_p)$. In view of the inequalities above, this is possible only when $\beta_i(M^n, R) = \beta_i(M^n, Z_p)$, which means that M^n has no torsion.

From now on we study the particular case of hypersurfaces ($N = 1$). Here the most important local invariant is the Gauss-Kronecker curvature $K(p)$ ($p \in M^n$). If $K(p) \neq 0$, the principal curvatures of M^n at p are all different from zero. In this case, we call the signature of M^n at p the nonnegative integer which is the excess of the number of principal curvatures of one sign over that of the opposite sign. A part of Theorem 3 is true for n dimensions, and we state it as a lemma:

LEMMA 2. *Let* $x: M^n \to E^{n+1}$ *be an immersion of a compact manifold as a hypersurface in Euclidean space with total curvature equal to the sum* $\beta(M^n)$ *of the Betti numbers of* M^n *relative to the coefficient field* mod 2. *Then every point* $p \in M^n$ *with* $K(p) > 0$ *and signature* n *lies on the outside of* M^n; *that is*, $x(M^n)$ *lies on one side of the tangent hyperplane at* $x(p)$.

To prove the lemma, we suppose the contrary, namely, that there is a point p with $K(p) > 0$ and signature n such that $x(M^n)$ lies on both sides of the tangent hyperplane at $x(p)$. Then there is a neighborhood U of p whose points have the same property. We orient U by choosing a field of unit normal vectors. The normal map $\nu: U \to S_0^n$ is then defined. Since $K(p) > 0$ ($p \in U$), ν can be supposed to be one-one (with the choice of a smaller neighborhood, if necessary). The image $\nu(U)$ is therefore of positive measure on S_0^n. It follows that there exists a set F of points of positive measure in $\nu(U)$ such that the function $\nu \cdot x(p)$ on M^n ($\nu \in F$) has only nondegenerate critical points, and such that there is a tangent hyperplane to $x(M^n)$ perpendicular to ν which divides $x(M^n)$ and which is tangent to $x(M^n)$ at a point of signature n. The latter is a critical point of index 0 or n for the function $\nu \cdot x(p)$, and it is neither a maximum nor a minimum. On the other hand, since the total curvature is equal to $\beta(M^n)$, and since $\nu \cdot x(p)$, for almost all $\nu \in S_0^n$, has at least β_i critical points of index i for each dimension i, the number of critical points of $\nu \cdot x(p)$ of indices 0 and n must be 1 ($= \beta_0 = \beta_n$), except for a set of points $\nu \in S_0^n$ of measure zero. These critical points are obviously the maximum and the minimum of the function $\nu \cdot x(p)$. Thus we arrive at a contradiction, and the lemma is proved.

We proceed to give a proof of Theorem 3. Half of the theorem follows from Lemma 2, because a point $p \in M^2$ with $K(p) > 0$ has the signature 2, the two principal curvatures being either both positive or both negative.

Suppose now that the surface M^2 is imbedded in E^3 in such a way that it lies on one side of the tangent plane at every point of positive Gaussian curvature. Suppose also that its total curvature is $> 2g + 2$. Then there exists a set of points $\nu \in S_0^2$ of positive measure such that the function $\nu \cdot x(p)$ on M^2 has only nondegenerate critical points, whose number exceeds $2g + 2$. Let m_i ($0 \leq i \leq 2$) be the number of critical points of index i of this function. Then we have by hypothesis

$$m_0 + m_1 + m_2 > 2g + 2,$$

and by Morse's relation, $m_0 - m_1 + m_2 = 2 - 2g$. Combination of these two relations gives $m_0 + m_2 > 2$. It follows that there are at least three distinct points of positive Gaussian curvature on M^2, whose tangent planes are perpendicular to ν. According to our hypothesis, two of these three tangent planes must coincide, and $x(M^2)$ is contained between the two tangent planes and is tangent to one of them, say π, in two distinct points. Since x is an imbedding, it is geometrically clear that we can rotate

π slightly so that the new plane is again tangent to $x(M^2)$ at a point of positive Gaussian curvature and divides $x(M^2)$. This contradiction proves Theorem 3.

Remark. Examples can easily be given to show that Theorem 3 is not true if x is an immersion.

3. HYPERSURFACES WITH NONNEGATIVE GAUSS-KRONECKER CURVATURE

As remarked before, an immersed compact orientable hypersurface with $K(p) > 0$ is convex. Its total curvature is equal to 2. We will show that the class of immersed compact hypersurfaces with $K(p) \geq 0$ is much wider.

LEMMA 3. *Let* $x: M^n \to E^{n+1}$ *be an immersion such that (1)* n *is even; (2)* M^n *is compact and orientable; (3)* $K(p) \geq 0$ $(p \in M^n)$. *Then* M^n *has no torsion, the odd-dimensional Betti numbers of* M^n *are zero, and its total curvature is equal to* $\beta(M^n)$.

As usual, let $\nu \in S_0^n$ be a unit vector such that the function $\nu \cdot x(p)$ on M^n has only nondegenerate critical points. The second-order terms in the expansion of the function at a critical point are given by $\nu \cdot d^2x = -d\nu \cdot dx$, which is the second fundamental form of the hypersurface. Since the critical points are nondegenerate, the Gauss-Kronecker curvature is > 0 at these points, and the numbers of negative principal curvatures and hence positive principal curvatures are both even. This means that the critical points of $\nu \cdot x(p)$ are of even indices. By the Theorem of Eilenberg and Shiffman stated in the footnote of Section 2, the manifold M^n is of the same homotopy type as a cell complex which consists only of even-dimensional cells. Hence the odd-dimensional Betti numbers of M^n are zero, and M^n has no torsion.

The degree of the normal map ν is equal to one-half of the Euler-Poincaré characteristic of M^n, which is in this case equal to $\beta(M^n)/2$. Since the image under ν of the set of points with $K(p) = 0$ is of measure zero, and since $K(p) > 0$ otherwise, the number of times which almost every point of S_0^n is covered by ν is $\beta(M^n)/2$. It follows that the total curvature of M^n is $\beta(M^n)$, because at every point of M^n there are two unit normal vectors, one being the negative of the other.

Remark. If besides the hypotheses of Lemma 3 we further suppose that the signature of M^n at p is equal to n at all points where $K(p) > 0$, then it follows that the Euler-Poincaré characteristic of M^n is $\beta(M^n)/2 = 1$. By Theorem 4 of TCI, we conclude that M^n is imbedded as a convex hypersurface.

Proof of Theorem 4. The first statement on compact orientable surfaces follows immediately from Lemma 3 and Theorem 4 of TCI. To prove the second statement, it suffices to exhibit some examples of hypersurfaces.

First let n be odd. In E^{n+1} with the coordinates $x^1, \cdots, x^{n+1}$, we consider the hypersurface with the equation

$$(r - 2)^2 + (x^{n+1})^2 = 1,$$

where

$$r^2 = (x^1)^2 + \cdots + (x^n)^2 \qquad (r \geq 0).$$

This hypersurface is obtained by rotating a unit circle about the x^{n+1}-axis, and is hence homeomorphic to the Cartesian product $S^1 \times S^{n-1}$. Its equation can also be written

$$x^{n+1} = \varepsilon\,\phi(r),$$

where $\varepsilon = \pm 1$ and

$$\phi(r) = +(1 - (r-2)^2)^{1/2}.$$

Then we have

$$dx = (dx^1, \cdots, dx^n, \varepsilon\, d\phi), \qquad d^2x = (0, \cdots, 0, \varepsilon\, d^2\phi),$$

$$\nu = \frac{\varepsilon}{+(1 + \phi_1^2 + \cdots + \phi_n^2)^{1/2}} (+\phi_1, \cdots, +\phi_n, -1),$$

where

$$\phi_i = \frac{\partial \phi}{\partial x^i}, \qquad \phi_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \qquad (1 \le i, j \le n).$$

This determination of ν is inward. It follows that

$$\nu \cdot d^2x = \frac{-1}{(1 + \phi_1^2 + \cdots + \phi_n^2)^{1/2}} d^2\phi = \frac{-1}{(1 + \phi_1^2 + \cdots \phi_n^2)^{1/2}} \sum_{i,j} \phi_{i,j} dx^i dx^j.$$

If ϕ' and ϕ'' denote the first and second derivatives of ϕ with respect to r, we have

$$\phi(r) = (-r^2 + 4r - 3)^{1/2}, \qquad \phi'(r) = \frac{2-r}{\phi(r)}, \qquad \phi''(r) = -\frac{1}{\phi(r)^3}\{\phi(r)^2 + (2-r)^2\},$$

and

$$\phi_i = \phi' \frac{x^i}{r}, \qquad \phi_{ij} = \phi'' \frac{x^i x^j}{r^2} + \frac{\phi'}{r^3}(\delta^{ij} r^2 - x^i x^j).$$

The Gauss-Kronecker curvature $K(p)$ is equal to the determinant of the second fundamental form divided by the determinant of the first fundamental form. Since the latter is positive, the sign of $K(p)$ is the same as that of $-\det(\phi_{ij})$. Since our hypersurface is a hypersurface of revolution, it suffices to consider those of its points in the (x^n, x^{n+1})-plane for which $x^1 = \cdots x^{n-1} = 0$. At such a point we have

$$-\det(\phi_{ij}) = -\frac{(\phi')^{n-1}}{r^{n-1}} \phi'' \ge 0.$$

The example for n even $(n \ge 4)$ is similar. It is a hypersurface obtained by rotating a two-dimensional sphere about a two-dimensional coordinate plane, and it has the equation

$$(r-2)^2 + (x^n)^2 + (x^{n+1})^2 = 1, \qquad r^2 = (x^1)^2 + \cdots + (x^{n-1})^2 \qquad (r \ge 0),$$

or

$$x^{n+1} = \varepsilon\,\psi(x^n, r) \qquad (\varepsilon^2 = 1),$$

where

$$\psi(x^n, r) = \{1 - (x^n)^2 - (r - 2)^2\}^{1/2} \geq 0.$$

As in the preceding example, K(p) has the same sign as $\det(\psi_{ij})$, where $\psi_{ij} = \frac{\partial^2 \psi}{\partial x^i \partial x^j}$. It is a straightforward computation to show that $\det(\psi_{ij}) \geq 0$ or $K(p) \geq 0$; we omit the details.

The following corollaries are obvious.

COROLLARY 1. *If a compact manifold* M *can be imbedded in* E^n, *then* $M \times S^n$ *can be imbedded in* E^{n+N}.

COROLLARY 2. *The product of spheres* $S^{n_1} \times \cdots \times S^{n_r}$ *can be imbedded in* $E^{n_1 + \cdots + n_r + 1}$ *with minimal total curvature* 2^r.

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