ON THE NUMERICAL RANGE OF A BOUNDED OPERATOR

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If T is a continuous linear transformation of a Hilbert space $\mathscr{H}$ into itself, its numerical range W(T) is defined as the set of all complex numbers (Tf, f) with $\|f\| = 1$. The most important facts about W(T) are the following.

(i) W(T) is convex [2, 4, 5, 6].

(ii) The closure of W(T) contains the spectrum of T [5].

(iii) If T is normal, the closure of W(T) is the smallest closed convex set containing the spectrum of T [5].

(iv) If T is normal and W(T) is closed, the extreme points of W(T) are eigenvalues [3].

(v) If W(T) reduces to the single point λ, then $T = \lambda I$, where I is the identity.

(vi) If W(T) is a subset of the real axis, T is self-adjoint.

In this note we obtain a precise description of W(T) for the special case where $\mathscr{H}$ is two-dimensional, and our other results follow very easily from this. Our methods also provide an easy and natural proof for (i), as well as alternative proofs for (v) and (vi).

The reduced angle between the two one-dimensional subspaces determined by f and g in $\mathscr{H}$ is that angle θ in the interval $0 \leq \theta \leq \pi/2$ for which

$$\cos \theta = \frac{|(f, g)|}{\|f\| \, \|g\|}.$$

Our point of departure is the following lemma; most of it is established in [6], but with a rather complicated proof.

LEMMA. *If* T *is a linear transformation of a two-dimensional Hilbert space into itself, its numerical range* W(T) *is an ellipse, the foci of which are the eigenvalues of* T. *If the eigenvalues are distinct, then the eccentricity of the ellipse is* $\sin \theta$, *where* θ *is the reduced angle between the eigenvectors; if there is only one eigenvalue* λ, *the ellipse is a circle with center* λ *and diameter* $\|T - \lambda\|$.

Proof. Since (Tf, f) is a continuous function on the compact surface of the unit sphere, it follows that W(T) is compact. We show that the boundary of W(T) is an ellipse of the required type, and that W(T) contains all interior points of that ellipse.

If T has only the eigenvalue λ, we let $S = T - \lambda I$ and note that $S^2 = 0$. If $S = 0$, then $T = \lambda I$, and W(T) contains only the point λ and is a circle with center λ and diameter $\|T - \lambda I\| = 0$. If S is not 0, there exists an orthonormal pair u and v for which $Su = 0$ and $Sv = \rho u$. It is immediate that $\|S\| = |\rho|$ and that W(S) is a circle of diameter $|\rho|$, hence that W(T) is a circle of diameter $\|T - \lambda I\|$ and center at λ.

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If T has two eigenvalues λ_1 and λ_2, we put $S = (T - \lambda_1 I)/(\lambda_2 - \lambda_1)$; the eigenvalues of S are 0 and 1, and $W(S)$ is obtained from $W(T)$ by a rigid motion and a homothetic transformation of the plane; thus $W(T)$ will have the asserted properties if $W(S)$ does. Let u and v be an orthonormal pair in the space for which $Su = 0$, and let w be the normalized eigenvector of S corresponding to the eigenvalue 1. Since w, u and v are only determined up to arbitrary multipliers of absolute value 1, we may choose them so that $w = (\cos\theta)u + (\sin\theta)v$, where θ is the reduced angle between the two eigenspaces of S (and therefore of T). We note that $Sv = (\cot\theta)u + v$, and we put $f = \zeta_1 u + \zeta_2 v$, where $|\zeta_1|^2 + |\zeta_2|^2 = 1$; then

$$(Sf, f) = |\zeta_2|^2 + \bar{\zeta}_1\zeta_2\cot\theta = |\zeta_2|^2 + |\zeta_1||\zeta_2|e^{i\omega}\cot\theta,$$

where ω depends only on the phase angles of ζ_1 and ζ_2. As ω varies, it is clear that we obtain a circle in the complex plane with center at the point $t = |\zeta_2|^2$ and with radius $\sqrt{t(1-t)}\cot\theta$. It follows that $W(S)$ is the union of the family of circles C_t, where $0 \leq t \leq 1$ and

$$C_t: \qquad (x - t)^2 + y^2 = (t - t^2)\cot^2\theta.$$

A straightforward computation then leads to the envelope of this family, namely the ellipse

$$\frac{\left(x - \frac{1}{2}\right)^2}{\left(\frac{\csc\theta}{2}\right)^2} + \frac{y^2}{\left(\frac{\cot\theta}{2}\right)^2} = 1$$

with the foci (0, 0) and (1, 0) and the eccentricity $\sin\theta$.

Any interior point λ_0 of this ellipse surely lies within the circle C_t which is tangent to the ellipse at the foot of the perpendicular from λ_0 to the ellipse, and it is exterior to one of the circles C_0 and C_1; since the circles vary continuously with t, it follows that there is a circle $C_{t'}$ through λ_0. Hence $W(T)$ is the closed ellipse, as desired.

Remark. If S is Hermitian, that is, if T is normal, then $\theta = \pi/2$ and the ellipse reduces to the line segment connecting the two eigenvalues.

The lemma leads to an immediate proof of the convexity of $W(T)$ in the most general case; if ζ_1 and ζ_2 are points of $W(T)$, there correspond normalized elements f_1 and f_2 in $\mathscr{H}$ for which $(Tf_k, f_k) = \zeta_k$. Let E be the projection on the two-dimensional subspace spanned by f_1 and f_2; it is readily seen that the numerical range $W(ETE)$ is contained in $W(T)$, and since the former set, an ellipse, contains the line segment connecting ζ_1 and ζ_2, the latter also contains this segment.

THEOREM 1. *If* $W(T)$ *is closed, then every boundary point at which the boundary is not a differentiable arc is an eigenvalue of* T.

Proof. From well-known results in the theory of convex functions, it follows that the boundary of $W(T)$ is differentiable except perhaps at enumerably many points. At each exceptional point there exists a left and a right tangent, and the angle between these tangents is smaller than π. Let λ be an exceptional boundary point, and f a normalized element of $\mathscr{H}$ such that $(Tf, f) = \lambda$. For any $g \neq 0$ in $\mathscr{H}$ we let E be the projection on the two-dimensional subspace spanned by f and g, and we consider the numerical range of ETE. This set is an ellipse contained in $W(T)$ and containing the point λ. Since no circle contained in $W(T)$ can pass through λ, it follows that the ellipse W(ETE) reduces to a line segment or a point. In either case, λ is an eigenvalue of ETE, and f is the corresponding eigenvector. Thus $Tf = \lambda f$; for if this were not so, we could choose $g = Tf$ and obtain a contradiction.

It should be observed that our theorem does not include Meng's theorem, that is, statement (iv) above; for that theorem makes all extreme points of W(T) eigenvalues when that set is closed and T is normal, whereas our theorem applies only to the "corners." The proof of (iv), like that of (iii), makes use of the spectral theorem for normal operators in an essential way.

THEOREM 2. *If a point* λ *of the spectrum of* T *is a boundary point of* W(T), *the transformation* T *has no elementary divisors associated with* λ.

Proof. If f is such that $(T - \lambda I)f \neq 0$ and $(T - \lambda I)^2 f = 0$, we consider the two-dimensional subspace spanned by f and $(T - \lambda I)f$, taking E as the projection upon it. By the lemma, W(ETE) contains a circle with center at λ and with nonzero radius, and this circle is contained in W(T), contrary to the hypothesis that λ is a boundary point.

The assertions (v) and (vi) are usually proved from the correspondance between quadratic forms and bilinear forms: the bilinear form is symmetric (respectively, 0) if and only if the quadratic form is real (respectively, 0) [1]. Simple alternative proofs for (v) and (vi) can be easily obtained from the lemma as follows: for every projection E with two-dimensional range, the transformation ETE is λE (respectively, self-adjoint), whence $T = \lambda I$ (respectively, whence T is self-adjoint).

REFERENCES

1. N. Aronszajn, *Introduction to the theory of Hilbert spaces*, Vol. 1, Stillwater, Oklahoma, 1950.

2. F. Hausdorff, *Der Wertvorrat einer Bilinearform*, Math. Z. 3 (1919), 314-316.

3. C. H. Meng, *A condition that a normal operator have a closed numerical range*, Proc. Amer. Math. Soc. 8 (1957), 85-88.

4. M. H. Stone, *Hausdorff's theorem concerning Hermitian forms*, Bull. Amer. Math. Soc. 36 (1930), 259-261.

5. ———, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloquium Publications 15 (1932).

6. O. Toeplitz, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. 2 (1918), 187-197.

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