A SIMPLIFIED PROOF OF THE PAPPUS-LEISENRING THEOREM

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The title refers to the following generalization to n dimensions of the projective theorem of Pappus:

THEOREM. *Let* S *be a commutative projective* n-*space* $(n > 1)$. *In a hyperplane* H_0 *of* S_n, *let* $T = \{t_i\}$ $(i = 0, 1, \cdots, n)$ *be a set of* $n + 1$ *points no proper subset of which are dependent. Let* $A_k^m = A_m^k$ $(k \neq n)$ *be the subspace determined by* $T - t_k - t_m$, *and through each* A_k^m *let there be passed two hyperplanes distinct from* H_0, *to be denoted by* H_k^m *and* H_m^k. *For each* k, *the* n *hyperplanes* H_k^m $(m = 0, 1, \cdots, k - 1, k + 1, \cdots, n)$ *determine a point* p_k. *Also, for each* m, *the* n *hyperplanes* H_k^m $(k = 0, 1, \cdots, m - 1, m + 1, \cdots, n)$ *determine a point* q_m. *If now the* p_k *are dependent, then so are the* q_m, *and the dependence is of the same rank.*

We shall simplify Leisenring's proof [1] by using an auxiliary point which introduces symmetry and thereby shortens the calculations. Since H_k^m contains p_k and q_m, the Grassmann products $G_k^m = (t_0 t_1 \cdots t_{k-1} p_k t_{k-1} \cdots t_{m-1} q_m t_{m+1} \cdots t_n)$ all vanish. (The order of k and m is not important.) By hypothesis also $(t_0 t_1 \cdots t_n) = 0$ and $(p_0 p_1 \cdots p_n) = 0$. We have to show that $(q_0 q_1 \cdots q_n) = 0$. Let w be any point not incident with H_0. We may write, for $i = (0, 1, \cdots, n)$,

$$p_i = \sum_{j=0}^{n} \lambda_{ij} t_j + w, \qquad \lambda_{jj} = 0, \tag{1}$$

$$q_i = \sum_{j=0}^{n} \mu_{ij} t_j + w, \qquad \mu_{jj} = 0, \tag{2}$$

$$0 = \sum_{j=0}^{n} t_j. \tag{3}$$

The last expression depends only on the choice of coordinate vectors for the t's. From the array one sees that the determinants for the p's and q's have the same form; for the q's, it is

$$\begin{vmatrix} 0 & \mu_{01} & \cdots & \mu_{0n} & 1 \\ \mu_{10} & 0 & \cdots & \mu_{1n} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}.$$

We shall show that in fact $\mu_{ji} = -\lambda_{ij}$; the theorem then follows on inspection of the determinant.

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By direct substitution from (1) and (2), we have

$$G_k^m = (t_0 t_1 \cdots t_{k-1}\left[\sum_{j=0}^{n} \lambda_{kj} t_j + w\right] t_{k+1} \cdots t_{m-1}\left[\sum_{j=0}^{n} \mu_{mj} t_j + w\right] t_{m+1} \cdots t_n) = 0.$$

Since $\lambda_{kk} = \mu_{mm} = 0$, this reduces at once to

$$(t_0 t_1 \cdots t_{k-1} [\lambda_{km} t_m + w] t_k \cdots t_{m-1} [\mu_{mk} t_k + w] t_{m+1} \cdots t_n) = 0.$$

In the expansion, the term containing both t_m and t_k vanishes, since all t's are present, and we have as the only nonzero terms

$$\lambda_{km} (t_0 t_1 \cdots t_{k-1} t_m t_k \cdots t_{m-1} w t_{m+1} \cdots t_n)$$
$$+ \mu_{mk} (t_0 t_1 \cdots t_{k-1} w t_{k+1} \cdots t_{m-1} t_k t_{m+1} \cdots t_n).$$

We now use (3), and in the second term we put $t_k = -\Sigma_{j=0}^{n} t_j$ $(j \neq k)$. This replaces t_k by $-t_m$, and the single interchange of t_m with w again changes the sign and makes the two parenthetic expressions identical. Since w is not in H_0, this product cannot vanish, and therefore $\lambda_{km} = -\mu_{mk}$.

In odd-dimensional spaces, a special situation arises: there exist analogues to Möbius tetrahedra (see [1], p. 40). If, in this case, we take but one hyperplane through each A_k^m (so that $p_i = q_i$), a computation like that above shows the determinant for the p's to be

$$\begin{vmatrix} 0 & \lambda_{01} & \cdots & \lambda_{0n} & 1 \\ -\lambda_{01} & 0 & \cdots & \lambda_{1n} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda_{0n} & -\lambda_{1n} & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix};$$

this is skew-symmetric and therefore vanishes if n is odd; that is, the p_k are necessarily dependent.

REFERENCE

1. K. B. Leisenring, *A theorem in projective* n*-space equivalent to commutativity,* Michigan Math. J. 2 (1953-54), 35-40.

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