ASYMPTOTIC SOLUTIONS WITH RESPECT TO A PARAMETER OF ORDINARY DIFFERENTIAL EQUATIONS HAVING A REGULAR SINGULAR POINT

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1. INTRODUCTION

We examine the asymptotic behavior for large $|\lambda|$ of solutions of the differential equation

$$\frac{d^2u}{dz^2} + [\lambda^2 q_0(z) + \lambda q_1(z) + F(z, \lambda)]u = 0, \tag{1.1}$$

with z restricted to a closed simply connected region $\mathfrak{D}$ containing the origin in its interior, and under the principal assumption that *the functions* zq_i *and* $z^2F(z, \lambda)$ *are analytic in* $\mathfrak{D}$. We also assume that $F(z, \lambda)$ *is analytic in* λ *when* $|\lambda| > N$. Thus

$$\text{(1.2)} \quad \begin{cases} \text{a)} \quad F(z, \lambda) = \sum_0^\infty f_i(z)\lambda^{-i} \quad (|\lambda| > N); \\ \text{and near the origin,} \\ \text{b)} \quad zq_i(z) = \sum_0^\infty q_{ik}z^k, \quad (i = 0, 1), \\ \text{c)} \quad z^2f_i(z) = \sum_0^\infty f_{ik}z^k. \end{cases}$$

We lose no generality by assuming that $q_{00} = 1$.

The main conclusions of the paper are Theorems 1, 2, and 3 in Sections 6 and 7. We observe, from these conclusions, that our theory is a special case of a general theory which also is applicable to equations of the type

$$\frac{d^2u}{dz^2} + \lambda^2 Q(z, \lambda)u = 0, \tag{1.3}$$

where $Q(z, \lambda) = \Sigma_0^\infty q_i(z)\lambda^{-i}$ and $q_0(z) = z^k \Sigma_0^\infty c_n z^n$ with $k = 0$ or $k = 1$. These equations were considered by Langer in [2]. Unfortunately, the general theory does not extend to values of k other than -1, 0, and 1.

The present work is motivated partly by the fact that the differential equations for several of the special functions, for example, the Whittaker, Legendre, and

Received January 29, 1957.

This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18(600)-1481.

Gaussian hypergeometric functions, are specializations of (1.1), and partly by a desire to take another step toward the completion of the theory of asymptotic solutions of equations of the type (1.3) that involve a turning point of order k ($k > -2$) at the origin. The fundamental character of the asymptotic representations of solutions of (1.1) has, in the case that $q_1(z) \equiv 0$, been obtained by Ziebur [4] and Olver [3]. Ziebur's work is further limited to a finite domain of z; and Olver assumes $F(z, \lambda)$ to be independent of λ, and $\arg \lambda$ to be fixed.

The first step in the investigation is the construction of a "related equation" which resembles (1.1) closely enough so that its solutions are asymptotic representations of solutions of (1.1). The derivation of this equation is based upon the work of Langer [2]. The procedure is familiar to the extent that details may be omitted here; the main line of argument is presented in Section 3. Once the related equation is constructed, we have only to establish the asymptotic representation of solutions of (1.1) by those of the related equation. The procedure followed to accomplish this in Sections 6 and 7 is again familiar (see for example [1]), and many details are omitted.

2. NOTATION

All hypotheses are in italics. The letters M and N are used as generic symbols for positive constants. A function of z and λ or of z, t, and λ which is uniformly bounded for $|\lambda| > N$ and z and t confined to $\mathfrak{D}$ or a specified subregion of $\mathfrak{D}$ is denoted by $O(1)$. In formulas in which the index j and the symbols $\pm$ or $\mp$ appear, the upper sign is to be used when $j = 1$, the lower when $j = 2$. The abbreviation $W(f_1, f_2, z)$ is used for the Wronskian $f_1(z) f_2'(z) - f_1'(z) f_2(z)$. The dash indicates differentiation with respect to z..

3. THE RELATED EQUATION

Let m be any nonnegative integer. We now derive a differential equation, to be called the related equation, which is identical with (1.1) to terms of order λ^{-m}, and whose solutions are explicitly known. This derivation involves three main steps: the construction of equations (3.3), (3.6), and (3.11), which are successively better approximations to equation (1.1).

We define $\phi(z)$ by the conditions

a) $$\phi^2 = q_0, \qquad \lim_{z \to 0} z^{1/2}\phi(z) = 1;$$

and in terms of ϕ we construct

(3.1)

b) $$\Phi(z) = \int_0^z \phi(t)\, dt,$$

c) $$\xi(z, \lambda) = \lambda \Phi(z),$$

d) $$\Psi(z) = [\phi(z)\, \Phi(z)]^{-1/2}, \qquad \Psi(0) = 2^{-1/2}.$$

The path of integration in $\int_0^z \phi(t)\, dt$ is not to encircle the origin. It is known [1] that if

$$v(z, \lambda) = \Psi(z)\xi\, C_\nu(\xi), \tag{3.2}$$

where $C_\nu(\xi)$ is a cylinder function of order ν, then v satisfies a first approximating equation

$$\frac{d^2v}{dz^2} + \left(\lambda^2 q_0 + \frac{(1-\nu^2)\phi^2}{\Phi^2} - \frac{\Psi''}{\Psi}\right)v = 0. \tag{3.3}$$

We assume that $\phi(z)$ *and* $\Phi(z)$ *are not zero for* $z \in \mathfrak{D}$, $z \neq 0$. The function $\Psi(z)$ is therefore analytic in $\mathfrak{D}$, and the coefficient of v in (3.3) is meromorphic in $\mathfrak{D}$, since q_0 and ϕ^2/Φ^2 have simple and double poles at $z = 0$, respectively.

Following Langer [2], we define

$$\begin{cases} \mu_0(z) = \cos\int_0^z \frac{q_1}{2\phi}\,dt, \qquad \mu_1(z) = \left(\sin\int_0^z \frac{q_1}{2\phi}\,dt\right)\phi^{-1}(z), \\ \theta(z,\nu) = (1-\nu^2)\phi^2\Phi^{-2} + \Psi''\Psi^{-1}, \\ D_0(z,\lambda) = 1 + \frac{\mu_0\mu_1' - \mu_0'\mu_1}{\lambda} + \frac{\theta\mu_1^2}{\lambda^2}. \end{cases} \tag{3.4}$$

The integrations are again to be taken along any convenient rectifiable path not encircling the origin. It is important to note that $\mu_0(z)$ and $\mu_1(z)$ are analytic in $\mathfrak{D}$, and that $\mu_1(z)$ has at least a simple zero at the origin. Thus even though $\theta(z,\nu)$ is singular at $z = 0$, D_0 is analytic in $\mathfrak{D}$. If $\mathfrak{D}$ is bounded, D_0 *is bounded away from zero in* $\mathfrak{D}$ *for* $|\lambda| > N$; if $\mathfrak{D}$ is not bounded, we assume this to be true. Now let

$$\zeta = D_0^{-1/2}[\mu_0 v + \mu_1 v'/\lambda], \tag{3.5}$$

where v is any solution of (3.3). Such a function ζ may be seen to satisfy the differential equation

$$\frac{d^2\zeta}{dz^2} + [\lambda^2 q_0 + \lambda q_1 + K^*(z,\lambda)]\zeta = 0, \tag{3.6}$$

with

$$\begin{cases} K^*(z,\lambda) = K_0 + D_0'' D_0^{-1} - \frac{3}{4}[D_0' D_0^{-1}]^2, \\ K_0(z,\lambda) = -D_0^{-1}[g_1(\mu_0 + \mu_1'\lambda^{-1}) + g_2(q_0\mu_1 - \mu_0'\lambda^{-1} + \theta\mu_1\lambda^{-2})], \\ g_1(z,\lambda) = \mu_0'' - \theta\mu_0 - (2\theta\mu_1' + \theta'\mu_1)\lambda^{-1}, \\ g_2(z,\lambda) = \mu_1'' - \theta\mu_1. \end{cases} \tag{3.7}$$

At $z = 0$, the function K_0 has a pole of order two whose coefficient is

$$D_0^{-1}(0,\lambda)\{\theta_0[1 + \mu_1'(0)\lambda^{-1} + (\mu_1'(0))^2\theta_0\lambda^{-2}]\}, \qquad \text{where } \theta_0 = (1-\nu^2)/4.$$

The part of this coefficient which is independent of λ is precisely θ_0. Thus we may choose the constant ν to be an analytic function of λ ($|\lambda| > N$) with nonnegative real part and having the property that $\Sigma_0^\infty f_{i0}$ is precisely the coefficient of z^{-2} in the Laurent expansion of $K^*(z, \lambda)$. (The numbers f_{i0} are defined by (1.2).) We do this and adopt the notation

$$K(z, \lambda) = K^*(z, \lambda) - z^{-2}\sum_0^\infty f_{i,0} = \sum_0^\infty k_j(z)\lambda^{-j}, \tag{3.8}$$

so that $zK(z, \lambda)$ is analytic for z in $\mathfrak{D}$ and $|\lambda| > N$.

The differential equation (3.6) resembles (1.1) up to terms which have at worst a simple pole at $z = 0$ and which are bounded in λ if $|\lambda| > N$. If no better approximation is desired, that is, if m is zero, the remaining step of this section may be omitted. To complete our construction of the related equation, we now define

$$\begin{cases} \eta = A(z, \lambda)\zeta + B(z, \lambda)\zeta'/\lambda^2, \\ A(z, \lambda) = \sum_0^{m-1} \alpha_j(z)\lambda^{-j}, \quad B(z, \lambda) = \sum_0^{m-1} \beta_j(z)\lambda^{-j}; \end{cases} \tag{3.9}$$

and we formally determine the functions α_i and β_i ($i = 0, \cdots, m - 1$) as in [2]. With the notation $h_i = \Sigma_1^\infty f_{ik} z^k$, this procedure leads to the following values for these coefficients:

$$\begin{cases} \alpha_0 = 1, \quad \alpha_1 = 0, \\ \alpha_j = -\frac{1}{2}\int_0^z \left[\beta_{j-2}'' + \sum_0^{j-2} (h_{j-s-2} - k_{j-s-2})\beta_s\right] dt \quad (j = 2, \cdots, m - 1), \\ \beta_0 = \frac{1}{\phi}\int_0^z \frac{h_0 - k_0}{2\phi} dt, \quad \beta_1 = \frac{1}{\phi}\int_0^z \frac{-2q_1\beta_0' - q_1'\beta_0 + h_1 - k_1}{2\phi} dt, \\ \beta_j = \frac{1}{\phi}\int_0^z \Big\{\alpha_j'' - 2q_1\beta_{j-1}' - q_1'\beta_{j-1} + \sum_0^j (h_{j-s} - k_{j-s})\alpha_s \\ \qquad - \sum_0^{j-2} [2k_{j-s-2}\beta_s' + k_{j-s-2}\beta_s + 2(t\beta_s' - \beta_s)t^{-3}f_{j-s-2,0}]\Big\}\frac{dt}{2\phi} \\ (j = 2, \cdots, m - 1). \end{cases} \tag{3.10}$$

The integrations are to be taken over paths which do not encircle the origin. The α_j and β_j are analytic in $\mathfrak{D}$, and each β_j has at least a simple zero at $z = 0$.

The functions η determined by (3.9), (3.10), and (3.5) satisfy a differential equation quite similar to (1.1) but possessing a term that involves η'. If we transform this equation by removing that term, we obtain the related equation

$$\frac{d^2y}{dz^2} + \left(\lambda^2 q_0 + q_1 + F(z, \lambda) + \frac{\Omega(z, \lambda)}{z\lambda^m}\right)y = 0, \tag{3.11}$$

in which $\Omega(z, \lambda)$ is a computable function of λ, q_0, q_1, and the $f_i(z)$. The specific form of $\Omega(z, \lambda)$ is of no interest to us; but it is crucial to note that $\Omega(z, \lambda)$ is bounded in λ for $|\lambda| > N$ and is analytic in $\mathfrak{D}$. The relations (3.9) and

$$(3.12) \qquad y = D_1^{-1/2}\eta, \qquad D_1 = \begin{vmatrix} A & B\lambda^{-2} \\ A' - q_0B - q_1B\lambda^{-1} - K^*B\lambda^{-2} & A + B'\lambda^{-2} \end{vmatrix}$$

give the connection between solutions of the related equation and the second approximating equation (3.6). The division by $D_1^{1/2}$ is legitimate if $|\lambda| > N$ and $\mathfrak{D}$ is bounded, since $D_1(z, \infty) = 1$. If $\mathfrak{D}$ is unbounded, we assume that D_1 *has no zeros in* $\mathfrak{D}$, *for* $|\lambda| > N$.

It is clear from the foregoing discussion and from [2], that if q_1 and $F(z, \lambda)$ are analytic at $z = 0$, the introduction of appropriate notation is all that is necessary to place this discussion in a framework which also includes the cases where q_0 does not vanish at the origin or has a simple zero there.

4. HYPOTHESES ON $\mathfrak{D}$

It is important to make a few further hypotheses concerning the mapping of $\mathfrak{D}$ by the function $\xi(z, \lambda)$. The complex parameter λ is to be chosen once for all with $|\lambda|$ sufficiently large to satisfy all requirements which have been or will be placed upon it. Since when $|z|$ is small, $\xi(z, \lambda) \approx 2\lambda z^{1/2}$, we shall consider $\mathfrak{D}$ to be a portion of a two-sheeted Riemann surface for which $c < \arg z \leq c + 4\pi$, c being any convenient real number. The function ξ maps $\mathfrak{D}$ upon a portion of a one-sheeted Riemann surface $\mathfrak{D}_\xi$. We assume that *the mapping from* $\mathfrak{D}$ *to* $\mathfrak{D}_\xi$ *is one-to-one.*

It is convenient to subdivide $\mathfrak{D}_\xi$ into a set of overlapping regions $\Xi^{(h)}$:

$$(4.1) \quad \Xi^{(h)} = \{\xi \mid \xi \in \mathfrak{D}_\xi \text{ and } (h - 1 + \varepsilon)\pi \leq \arg \xi \leq (h + 1 - \varepsilon)\pi\} \qquad (h = 0, \pm 1, \cdots),$$

where ε is positive and sufficiently small. The admissible values of h of course depend upon the range of $\arg \xi$ in $\mathfrak{D}_\xi$ and will always be finite in number. Our final hypotheses on $\mathfrak{D}$ are these:

i) *For each region* $\Xi^{(h)} \subset \mathfrak{D}_\xi$ *there exists a point* z_{+M} $[z_{-M}]$ *in the image of* $\Xi^{(h)}$ *on* $\mathfrak{D}$ *such that all points of* $\Xi^{(h)} \cup \{\xi \mid |\xi| \leq N\}$ *may be joined to* $\xi(z_{+M})$ $[\xi(z_{-M})]$ *by an arc* Γ *directed from* $\xi(z_{+M})$ $[\xi(z_{-M})]$ *to* $\xi(z, \lambda)$ *and having the following properties: along the part of* Γ *lying outside* $|\xi| = N$, $\Im\xi$ *is monotone decreasing* [*increasing*] *and* $\xi \in \Xi^{(h)}$; *and the remaining part, if any, consists of an arc of* $|\xi| = N$ *and a segment of the radius of* $|\xi| = N$ *leading to* $\xi(z, \lambda)$.

ii) *The origin may be joined to any point of* $\mathfrak{D}_\xi$ *by an arc* Γ *which lies in the same region* $\Xi^{(h)}$ *as* $\xi(z)$ *and along which* $\Im\xi$ *is monotonic.* (The image of a *path* Γ on $\mathfrak{D}$ may without confusion be denoted by the same symbol Γ.)

iii) *The integrals*

$$\int_{\Gamma} \frac{\Omega\Phi\, dt}{t\gamma^2 D_0 D_1}$$

(for the definition of $\gamma(z)$, see (6.5)) *are uniformly bounded with respect to all paths* Γ *contained in the subregion of* $\mathfrak{D}$ *for which* $|z| > N$. (If $\mathfrak{D}$ is bounded, this hypothesis is automatically fulfilled).

5. SOLUTIONS OF THE RELATED EQUATION

We now single out for future use certain pairs of linearly independent solutions of (3.11). These are chosen for the simplicity of their behavior at $\xi = 0$ and at $\xi = \infty$. It follows from the several relations (3.12), (3.9), (3.5), and (3.2) that any solution of the related equation (3.11) has the form

$$y = [E_0\Psi + \lambda^{-1}E_1\Psi']\xi C_\nu(\xi) + E_1\phi\Psi[\xi C_{\nu-1}(\xi) + (1-\nu)C_\nu(\xi)], \tag{5.1}$$

where

$$\begin{cases} E_0 = (D_0D_1)^{-1/2}\left[A\mu_0 - \dfrac{D_0'B\mu_0}{2\lambda^2 D_0} + \dfrac{B}{\lambda^2}\left(\mu_0' - \lambda q_0\mu_1 - \dfrac{\theta\mu_1}{\lambda}\right)\right], \\[2ex] E_1 = (D_0D_1)^{-1/2}\left[\left(A - \dfrac{D_0'B}{2\lambda^2 D_0}\right)\mu_1 + \dfrac{B}{\lambda}\left(\mu_0 + \dfrac{\mu_1'}{\lambda}\right)\right]. \end{cases} \tag{5.2}$$

Both of the functions E_j are analytic in $\mathfrak{D}$, and E_1 has at least a simple zero at $z = 0$.

A pair y_1 and y_2 of linearly independent solutions of (3.11) is determined by (5.1) when $C_\nu = J_\nu$ and $C_\nu = Y_\nu$, respectively. If $|\xi(z, \lambda)| \leq N$, then

$$\begin{cases} y_1(z, \lambda) = \xi^{1+\nu}O(1), \quad y_2(z, \lambda) = \xi^{1-\nu}O(1) \quad (\Re\nu > 0), \\ y_2(z, \lambda) = \xi[O(1)\ln\xi + O(1)] \quad (\Re\nu = 0). \end{cases} \tag{5.3}$$

Also, because y is a linear form in v and v',

$$W(y_1, y_2, z) = W(\Psi\xi J_\nu(\xi), \Psi\xi Y_\nu(\xi), z) = 2\lambda^2/\pi.$$

When large values of $|\xi|$ are in question, the solutions obtained from (5.1) by using Bessel functions of the third kind are advantageous. We let $y_{2n,1}$ and $y_{2n,2}$ denote the solutions determined by (5.1) when

$$C_\nu = \left(\frac{\pi}{2}\right)^{1/2}\exp[(\nu + 2n + 1/2)\pi i/2]H_\nu^{(1)}(\xi e^{-2n\pi i})$$

and

$$C_\nu = \left(\frac{\pi}{2}\right)^{1/2}\exp[-(\nu + 2n + 1/2)\pi i/2]H_\nu^{(2)}(\xi e^{-2n\pi i})$$

respectively; and we define

$$y_{2n+1,1}(z) \equiv y_{2n,1}(z), \qquad y_{2n-1,2}(z) \equiv y_{2n,2}(z).$$

Each integral index k is thus associated with a pair of solutions y_{kj} $(j = 1, 2)$. It is easily shown that

$$W(y_{k1}, y_{k2}, z) = -2i\lambda^2.$$

The Hankel functions $H_\nu^{(j)}(e^{-2n\pi i}\xi)$ (j = 1, 2) have the following property, for $|\xi| > N$ and $\xi \in \Xi^{(h)}$ (provided h = 2n or h = 2n + 1 if j = 1, and h = 2n or h = 2n - 1 if j = 2):

$$(5.4) \qquad H_\nu^{(j)}(e^{-2n\pi i}\xi) \sim \left(\frac{2}{\pi\xi}\right)^{1/2} \exp\left[\pm i\left(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] {}_2F_0\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \frac{\pm 1}{2i\xi}\right).$$

Consequently, the solutions y_{kj} defined above have a simple asymptotic behavior in $\Xi^{(k)}$, namely,

$$(5.5) \qquad \begin{aligned} y_{2n,j}(z, \lambda) &= [E_0\Psi + \lambda^{-1} E_1\Psi']\xi^{1/2}e^{\pm i\xi}[1 + \xi^{-1}O(1)] \\ &\pm E_1\phi\Psi i\xi^{1/2}e^{\pm i\xi}[1 + \xi^{-1}O(1)], \qquad \text{for } \xi \in (\Xi^{(2n)} \cup \Xi^{(2n\pm 1)}). \end{aligned}$$

These asymptotic forms may be differentiated with respect to z. Using (5.4) in its full generality, we may make them explicit to terms of order $\xi^{-m-1}O(1)$. When ξ is not confined to the regions specified in (5.5), the behavior of the solutions y_{kj} may be found from the formula

$$(5.6) \qquad \begin{cases} \text{a)} & y_{2n,j} = c_{j,1}^{n,k} y_{k,1} + c_{j,2}^{n,k} y_{k,2} \quad (\xi \in \Xi^{(k)}), \text{ where [1, p. 404]} \\ \text{b)} & c_{j,1}^{n,2s} = (-1)^{n-s+1} i^{j-1} \dfrac{\sin(2s - 2n + j - 2)\pi\nu}{\sin \pi\nu}, \\ \text{c)} & c_{j,2}^{n,2s} = (-1)^{n-s+j} i^{2-j} \dfrac{\sin(2s - 2n + j - 1)\pi\nu}{\sin \pi\nu}, \\ \text{d)} & c_{j,2}^{n,2s+1} = c_{j,2}^{n,2s}, \quad c_{j,1}^{n,2s-1} = c_{j,1}^{n,2s}. \end{cases}$$

Also,

$$(5.7) \qquad \begin{cases} y_1 = c_1^{(k)} y_{k1} + c_2^{(k)} y_{k2} \quad (\xi \in \Xi^{(k)}), \text{ where} \\ c_j^{(2s)} = (2\pi)^{-1/2} e^{(2s\mp 1/2)(\nu\pm 1/2)\pi i}, \quad c_j^{(2s\pm 1)} = c_j^{(2s)} \quad (j = 1, 2). \end{cases}$$

6. THE BOUNDED SOLUTION OF (1.1) WHEN $|\xi| \leq N$

We are now in a position to demonstrate that solutions of the related equation (3.11) are asymptotic representations for solutions of (1.1). We compare solutions of these two equations by means of the integral equation

$$(6.1) \qquad u = y - \int_{z_*}^{z} \frac{y_a(z)\,y_b(t) - y_a(t)\,y_b(z)}{W(y_a, y_b, t)} \cdot \frac{\Omega(t, \lambda)\,u(t)}{t\lambda^m}\,dt,$$

which is equivalent to (1.1). In this equation, y_a and y_b may be any pair of linearly independent solutions of (3.11), y may be any solution, and z_* may be any point of $\mathfrak{D}$. The kernel in (6.1) is, of course, independent of the choice of the pair y_a and y_b.

We first consider that solution of (6.1) which vanishes at the origin to a higher order than any which is linearly independent of it. This solution, $u_1(z, \lambda)$, has exponent $(1 + \nu)/2$ at the origin and is determined by the choices $y = y_1$ and $z_* = 0$ in (6.1). We treat the cases $|\xi| \leq N$ and $|\xi| > N$ separately.

I. $|\xi| \leq N$. In (6.1) we choose $y_a = y_1$ and $y_b = y_2$, and we choose the path of integration to be the image on $\mathfrak{D}$ of the straight line from 0 to $\xi(z)$. Let $\xi(t) = \tau$, and assume for the moment that $\Re\nu > 0$. Define $Y_j = \xi^{-1\mp\nu} y_j$ (j = 1, 2), and $U_1 = \xi^{-1-\nu} u_1$ (we use this notation in this paragraph only). The integral equation (6.1) may now be given the form

$$U_1(z) = Y_1(z) - \frac{\pi}{2}\int_0^z [Y_1(z)\, Y_2(t) - Y_1(t)\, Y_2(z)\, (\tau/\xi)^2] \frac{\tau^2 \Omega(t)\, U_1(t)}{t\lambda^{m+2}}\, dt,$$

which we abbreviate to

$$U_1(z) = Y_1(z) + \lambda^{-m-2}\int_0^z K(z, t)\, U_1(t)\, dt.$$

With

$$Y_1^{(0)}(z) = Y_1(z) \qquad \text{and} \qquad Y_1^{(n)}(z) = \lambda^{-m-2}\int_0^z K(z, t)\, Y_1^{(n-1)}(t)\, dt,$$

the familiar process of iteration leads formally to the relation

$$U_1(z) = \sum_0^\infty Y_1^{(n)}(z). \tag{6.2}$$

We demonstrate the uniform convergence of the series on the right of (6.2) for $|\xi| \leq N$, and thereby establish (6.2). The relations (5.3) and the fact that $|\tau| \leq |\xi|$ imply that the quantities Y_j and $(\tau/\xi)^{2\nu}$ are bounded. Since

$$\tau^2 \approx 4\lambda^2 t, \quad dt = (\lambda\phi)^{-1} d\tau = \lambda^{-2}\, O(1)\, \tau\, d\tau.$$

Thus

$$\left|\int_0^z K(z, t)\, Y_1(t)\, dt\right| \leq M\int_0^\xi |\tau\, d\tau| < \frac{M}{2}|\xi|^2.$$

Therefore,

$$\left|Y_1^{(1)}(z)\right| < \frac{M|\xi|^2}{2|\lambda|^{m+2}}.$$

The fact that

$$\left|Y_1^{(n)}(z)\right| < \frac{M^n |\xi|^{2n}}{(n+1)!\, |\lambda|^{(m+2)n}} \qquad (n = 1, 2, \cdots)$$

for $|\xi| \leq N$ may now be easily established by induction. It follows that (6.2) holds if $|\xi| \leq N$ and $\Re\nu > 0$. If $\Re\nu = 0$, then

$$K(z, t) = O(1) \ln\tau + O(1) \ln \xi + O(1),$$

and

$$\left| Y_1^{(1)}(z) \right| < \frac{M|\xi|^2 |\ln |\xi||}{|\lambda|^{m+2}}.$$

Consequently,

$$\left| Y_1^{(n)}(z) \right| < \frac{M^n |\xi|^{(2-\varepsilon)n}}{|\lambda|^{(m+2)n}} \qquad (n = 2, 3, \cdots),$$

for any ε $(0 < \varepsilon < 1)$, so that (6.2) again holds. Thus, when $|\xi(z, \lambda)| \leq N$,

$$\text{(6.3)} \qquad \begin{cases} u_1(z, \lambda) = y_1(z, \lambda) + \dfrac{O(1)\, \xi^{3+\nu}}{\lambda^{m+2}} & \text{if } \Re\nu > 0, \\ u_1(z, \lambda) = y_1(z, \lambda) + \dfrac{O(1)\, \xi^3 \ln \xi}{\lambda^{m+2}} & \text{if } \Re\nu = 0. \end{cases}$$

These relations show that u_1 vanishes to a higher order at $z = 0$ than any other solution linearly independent of it.

Differentiating (6.1) with respect to z and substituting the estimates (6.3) for u_1 in the derived equation, we come to the conclusion that

$$\text{(6.4)} \qquad \begin{cases} u_1'(z, \lambda) = y_1'(z, \lambda) + \dfrac{O(1)\, \xi^{1+\nu}}{\lambda^m} & (\Re\nu > 0), \\ u_1'(z, \lambda) = y_1'(z, \lambda) + \dfrac{O(1)\, \xi \ln \xi}{\lambda^m} & (\Re\nu = 0). \end{cases}$$

II. $|\xi| > N$. When $|\xi|$ is large, either the exponential $e^{i\xi}$ or the exponential $e^{-i\xi}$ causes $|y_1|$ to be large, according as $\Im z$ is negative or positive. The deductions for u_1 are now to be based upon (5.7), (5.6), and (5.5); and they must be appropriately adapted to the location of ξ. We give a full discussion only for $\Im \xi \leq 0$, which we momentarily assume to be the case. Let $\gamma(z)$ be any convenient nonzero analytic function having the property that for each k the functions

$$\text{(6.5)} \qquad \begin{cases} Y_j = (D_0 D_1)^{1/2} \gamma(z)\, \xi^{-1/2} e^{-i\xi} y_j, \\ Y_{kj} = (D_0 D_1)^{1/2} \gamma(z)\, \xi^{-1/2} e^{\mp i\xi} y_{kj} \end{cases} \qquad (j = 1, 2),$$

are bounded in $\Xi^{(k)} \cap \{\xi \,|\, |\xi| > N\}$. If $\mathfrak{D}$ is a bounded domain, $\gamma(z)$ may be chosen to be 1.

The integration in (6.1) is now taken over a path Γ extending from 0 to z. For the consideration of the integral, we divide Γ into three parts: Γ_1, a radius of the circle $|\tau| = N$; Γ_2, a rectifiable arc on which $|\tau| > N$; Γ_3, the remaining arc, if any. If $\mathfrak{D}$ is finite, no arc Γ_3 need be considered. We may now rewrite (6.1) in the form

$$\text{(6.6)} \qquad U_1(z) = Y_1(z) + \sum_1^3 \int_{\Gamma_i} K_i(z, t)\, U_1(t)\, dt,$$

where

$$U_1(z) = (D_0D_1)^{1/2}\gamma(z)\,\xi^{-1/2}e^{-i\xi}u_1(z),$$

$$K_1(z,t) = -\frac{\pi}{2}\,[Y_1(z)\,y_2(t) - y_1(t)\,Y_2(z)]\,\frac{e^{i\tau}\tau^{1/2}\Omega(t)}{(D_0D_1)^{1/2}t\lambda^{m+2}},$$

$$K_2(z,t) = -\frac{i}{2}\,[Y_{k1}(z)\,Y_{k2}(t) - Y_{k1}(t)\,Y_{k2}(z)e^{2i(\tau-\xi)}]\frac{\Omega\,\tau}{t\gamma^2 D_0D_1\lambda^{m+2}},$$

$$K_3(z,t) = K_2(z,t).$$

The estimates used in the discussion of Case I, $|\xi| \leq N$, lead to the conclusion that

$$\int_{\Gamma_1} K_1(z,t)\,Y_1(t)\,dt = \lambda^{-m-2}\,O(1).$$

On $\Gamma_2 \cup \Gamma_3$, $\Im\tau \geq \Im\xi$. On Γ_2, $|\tau|$ is bounded; hence, the quantities Y_{kj} and $\tau\Omega(\lambda\gamma^2 tD_0D_1)^{-1}$ are bounded. Thus

$$\int_{\Gamma_2} K_2(z,t)\,Y_1(t)\,dt = \lambda^{-m-1}\,O(1).$$

In virtue of the hypothesis (iii) of Section 4, we may also conclude that

$$\int_{\Gamma_3} K_3(z,t)\,Y_1(t)\,dt = \lambda^{-m-1}\,O(1).$$

From these facts it follows by induction that the formal iteration of equation (6.6) is legitimate. Consequently, when $|\xi| > N$ and $\Im\xi \leq 0$,

$$U_1(z) = Y_1(z) + \lambda^{-m-1}O(1).$$

The form of u_1 when ξ lies in an upper half-plane differs from that just derived only in that the roles of $e^{i\xi}$ and $e^{-i\xi}$ are interchanged. Thus, for all ξ with $|\xi| > N$,

$$u_1(z,\lambda) = y_1(z,\lambda) + \frac{\xi^{1/2}[e^{i\xi}O(1) + e^{-i\xi}O(1)]}{\gamma\,(D_0D_1)^{1/2}\lambda^{m+1}}. \tag{6.7}$$

The form of u_1' may again be found by differentiating (6.1). It is

$$u_1'(z,\lambda) = y_1'(z,\lambda) + \frac{e^{-2i\xi}y_{k1}'(z,\lambda)\,O(1) + e^{2i\xi}y_{k2}'(z,\lambda)\,O(1)}{\lambda^{m+1}}, \tag{6.8}$$

provided $|\xi| > N$ and $\xi \in \Xi^{(k)}$.

We summarize the conclusions which we have thus far obtained:

THEOREM 1. *Under the hypotheses set forth in Sections* 1, 3, *and* 4, *the solution of the differential equation* (1.1) *with exponent* $(1+\nu)/2$ *at* $z = 0$, *together with its first* *derivative, is described by* (6.3) *and* (6.4) *when* $|\xi| \leq N$, *and by the forms* (6.7) *and* (6.8) *when* $|\xi| > N$. The functions y_j and y_{kj} $(j = 1, 2)$, appearing in these formulas are defined in Section 5; the functions ξ, D_0, D_1, and γ and the regions $\Xi^{(k)}$ are defined by the relations (3.1), (3.4), (3.12), (6.5), and (4.1), respectively.

7. SUBDOMINANT SOLUTIONS OF (1.1)

Theorem 1 reveals that while $u_1(z, \lambda) \to 0$ as $z \to 0$ for fixed λ, it becomes infinite as $|\lambda| \to \infty$ for each $z \in \mathfrak{D}$ with $\Im(\xi(z)) \neq 0$ and $z \neq 0$; that is, u_1 is dominant almost everywhere in $\mathfrak{D}$. There are, however, solutions of (1.1) which are bounded in λ over certain subregions of $\mathfrak{D}$. The form of these subdominant solutions will now be described.

Let the integral equation (6.1) be written with the roles of y, u, and z_* taken by y_{k2}, u_{k2}, and z_{+M} or y_{k1}, u_{k1}, and z_{-M}, respectively. The analysis leading to the description of the solutions u_{kj} when $|\xi| > N$ resembles that which precedes Theorem 1, and it will be omitted. The conclusions are:

THEOREM 2. *Under the hypotheses set forth in Sections* 1, 3, *and* 4 *and corresponding to each admissible region* $\Xi^{(k)}$ *defined by* (4.1), *the differential equation* (1.1) *has a pair of linearly independent solutions* u_{kj} $(j = 1, 2)$ *such that when* $|\xi| > N$,

$$(7.1a)\quad \begin{cases} u_{2n+1,1}(z, \lambda) \equiv u_{2n,1}(z, \lambda) = y_{2n,1}(z, \lambda) + \dfrac{\xi^{1/2} e^{i\xi} O(1)}{\gamma (D_0 D_1)^{1/2} \lambda^{m+1}}, \\ u'_{2n,1}(z, \lambda) = y'_{2n,1}(z, \lambda) + \dfrac{y'_{2n,1}(z, \lambda) O(1) + y'_{2n,2}(z, \lambda) e^{2i\xi} O(1)}{\lambda^{m+1}}, \end{cases}$$

for $\xi \in (\Xi^{(2n)} \cup \Xi^{(2n+1)})$, *and*

$$(7.1b)\quad \begin{cases} u_{2n-1,2}(z, \lambda) \equiv u_{2n,2}(z, \lambda) = y_{2n,2}(z, \lambda) + \dfrac{\xi^{1/2} e^{-i\xi} O(1)}{\gamma \cdot (D_0 D_1)^{1/2} \lambda^{m+1}}, \\ u'_{2n,2}(z, \lambda) = y'_{2n,2}(z, \lambda) + \dfrac{y'_{2n,2}(z, \lambda) O(1) + y'_{2n,1}(z, \lambda) O(1) e^{-2i\xi}}{\lambda^{m+1}}, \end{cases}$$

for $\xi \in (\Xi^{(2n-1)} \cup \Xi^{(2n)})$. The functions y_{kj} are described in Section 5; the functions ξ, D_0, D_1, and γ are defined by the relations (3.1), (3.4), (3.12), and (6.5), respectively.

These results show that each solution u_{kj} is bounded as $|\lambda| \to \infty$, when ξ lies in an upper or lower half-plane of $\Xi^{(k)}$ according as $j = 1$ or $j = 2$.

We next determine the behavior of u_{kj} when $|\xi| \leq N$. The results obtained will enable us to find the form of the solutions u_{kj} when $|\xi| > N$ while ξ is not restricted to lie in one of the subregions of $\mathfrak{D}$ indicated in Theorem 2.

We consider the integral equation (6.1) with the roles of y, u, and z_* taken by $y_{2n,2}$, $u_{2n,2}$, and z_{-M}. The path of integration is to be a curve Γ. The path may be divided into two parts: Γ_1, a portion of a radius of $|\tau| = N$; and Γ_2, the remainder of the curve. On the arc Γ_2, Theorem 2 provides an estimate for $u_{2n,2}$. Making use of this and the hypothesis (iii) of Section 4, we can show that the part of the integral in (6.1) taken over Γ_2 has the form

$$y_{2n,1}(z)\,O(1)\,\lambda^{-m-1} + y_{2n,2}(z)\,O(1)\,\lambda^{-m-1},$$

or

$$\xi^{-1-\nu}O(1)\,\lambda^{-m-1} \quad \text{if } \Re\nu > 0, \qquad O(1)\lambda^{-m-1}\,\xi \ln \xi \quad \text{if } \Re\nu = 0.$$

Thus, if $\Re\nu > 0$,

$$u_{2n,2}(z) = y_{2n,2}(z) + \frac{\xi^{1-\nu}O(1)}{\lambda^{m+1}} - \frac{\pi}{2}\int_{\Gamma_1} [y_1(z)\,y_2(t) - y_1(t)\,y_2(z)]\frac{\Omega(t)\,u_{2n,2}(t)}{t\lambda^{m+2}}\,dt;$$

and an alternative expression holds if $\Re\nu = 0$.

We assume, for the moment, that $\Re\nu$ is positive, and we define

$$U_{2n,2} = \xi^{-1+\nu}u_{2n,2}, \qquad Y_{2n,2} = \xi^{-1+\nu}y_{2n,2}, \qquad Y_j = \xi^{-1\mp\nu}y_j.$$

Using these abbreviations, we may write the last equation in the form

$$U_{2n,2}(z) = Y_{2n,2}(z) + \frac{O(1)}{\lambda^{m+1}} + \int_{\Gamma_1} K(z, t)\,U_{2n,2}(t)\,dt, \tag{7.2}$$

where

$$K(z, t) = -\frac{\pi}{2}[Y_1(z)\,Y_2(t)\,(\xi/\tau)^{2\nu} - Y_1(t)\,Y_2(z)]\frac{\tau^2\,\Omega(t)}{t\lambda^{m+2}}.$$

On Γ_1, the functions Y_j and Ω are bounded; and since $|\tau| \geq |\xi|$, $(\xi/\tau)^{2\nu}$ is bounded. Also,

$$\int_{\Gamma_1} \frac{\tau^2\,dt}{t\lambda^{m+2}} = \frac{O(1)}{\lambda^{m+2}}.$$

Therefore,

$$\int_{\Gamma_1} K(z, t)\,[Y_{2n,2}(t) + O(1)\lambda^{-m-1}]\,dt = O(1)\lambda^{-m-2}.$$

We may now proceed to solve (7.2) by iteration and to show that

$$U_{2n,2}(z) = Y_{2n,2}(z) + O(1)\lambda^{-m-1}, \qquad \text{for } |\xi| \leq N \text{ and } \Re\nu > 0.$$

An analogous result holds when $\Re\nu = 0$. Moreover, a similar discussion applies to the solution $u_{2n,1}(z, \lambda)$. Consequently, when $|\xi| \leq N$,

$$(7.3)\qquad \begin{cases} u_{kj}(z, \lambda) = y_{kj}(z, \lambda) + \dfrac{\xi^{1-\nu}O(1)}{\lambda^{m+1}} & (\Re\nu > 0;\ j = 1, 2), \\ u_{kj}(z, \lambda) = y_{kj}(z, \lambda) + \dfrac{O(1)\,\xi\,\ln\xi}{\lambda^{m+1}} & (\Re\nu = 0;\ j = 1, 2). \end{cases}$$

The insertion of these expressions in the differentiated form of (6.1) yields the formulas

$$(7.4)\qquad \begin{cases} u'_{kj}(z, \lambda) = y'_{kj}(z, \lambda) + \dfrac{\xi^{-1-\nu}O(1)}{\lambda^{m-1}} & (\Re\nu > 0), \\ u'_{kj}(z, \lambda) = y'_{kj}(z, \lambda) + \dfrac{O(1)\,\xi^{-1}\ln\xi}{\lambda^{m-1}} & (\Re\nu = 0). \end{cases}$$

It remains to give a description of the behavior of the solutions u_{kj} when $|\xi| > N$ and ξ is not confined to a specific subregion of $\mathfrak{D}_\xi$ as indicated in (7.1). Now, for $z \neq 0$, the solutions $u_{hj}(z, \lambda)$ may be related to any other pair $u_{2k,j}(z, \lambda)$ by the formulas

$$(7.5)\qquad u_{2k,j}(z, \lambda) = c_{j1}(\lambda)\,u_{h1}(z, \lambda) + u_{h2}(z, \lambda) \qquad (j = 1, 2).$$

The relations (5.6) give the analogous formulas for $y_{2k,j}$ in terms of y_{h1} and y_{h2}. Let us replace the u's in (7.5) by their asymptotic forms as given by (7.3). If we then replace the term y_{kj} in the resulting expression by its equivalent in (5.6), we find that

$$\left(c_{j,1} - c_{j,1}^{(k,h)}\right)y_{h1} + \left(c_{j,2} - c_{j,2}^{(k,h)}\right)y_{h,2} = \begin{cases} O(1)\,\xi^{+1-\nu}\,\lambda^{-m-1} & (\Re\nu > 0), \\ O(1)\,\lambda^{-m-1}\,\xi\,\ln\xi & (\Re\nu = 0). \end{cases}$$

Since, near $z = 0$, $y_{hj} = \xi^{+1-\nu}O(1)$ or $y_{hj} = O(1)\,\xi\,\ln\xi$ according as $\Re\nu$ is positive or zero, it follows from this equation that

$$c_{j,i} = c_{j,i}^{(k,h)} + O(1)\,\lambda^{-m-1}.$$

These values for the coefficients $c_{j,i}$ may now be used in (7.5), with the result that

$$(7.6)\qquad u_{2k\pm1,j}(z, \lambda) \equiv u_{2k,j}(z, \lambda) = y_{2k,j}(z, \lambda) + \frac{\xi^{1/2}[e^{i\xi}O(1) + e^{-i\xi}O(1)]}{\gamma\,(D_0D_1)^{1/2}\,\lambda^{m+1}} \qquad (j = 1, 2)$$

when $|\xi| > N$. These formulas are valid without restriction on arg ξ. It should be noted that the formulas (7.1) are more precise, when they apply. Forms for u'_{kj} when arg ξ is not restricted may be obtained by differentiating (7.5). The summary of the conclusions reached relative to the solutions u_{kj} is:

THEOREM 3. *Under the hypotheses set forth in Sections* 1, 3, *and* 4, *the differential equation* (1.1) *has pairs of linearly independent solutions* $u_{kj}(z, \lambda)$ $(j = 1, 2;$ $k = 0, \pm1, \cdots)$ (*the admissible values of* k *depending upon the range of* arg ξ *in* $\mathfrak{D}_\xi$) *which are described by* (7.3) *and* (7.4) *if* $|\xi| \leq N$; *and which are generally described by* (7.6) *if* $|\xi| > N$, *but more precisely by* (7.1) *when the latter formulas are applicable.*

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