ON THE EMBEDDING OF A GENERALIZED REGULAR RING IN A RING WITH IDENTITY

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1. It is well known that every ring may be embedded in a ring with identity, by a process (to be described below) which was apparently first discussed by Dorroh (see [1]). In the earlier of Stone's famous papers on Boolean rings it is shown that, by use of a suitable special case of this process, each Boolean ring may be embedded in a Boolean ring with identity which is minimal in the class of Boolean rings with identity containing the given ring [5, Theorem 1]. Brown and McCoy extended this result to p-rings [1, Corollary 1 to Theorem 5]. (In fact, they showed that the corresponding extension is minimal in the class of rings of characteristic p.)

We consider here the embedding problem for commutative regular rings and their generalizations. In particular, we find under what conditions a commutative semi-simple ring which is regular, m-regular or π-regular may be embedded in a commutative ring with identity of the same type. (In the case of a regular ring, the assumption of semi-simplicity is, of course, superfluous.) The minimality question will not be taken up, however.

2. We shall confine our attention entirely to commutative rings, merely remarking that some of the material which follows could be presented without the requirement of commutativity.

A commutative ring A is said to be: (1) *regular* if, for each $a \in A$, there is an $x \in A$ satisfying $a^2x = a$; (2) m-*regular* if there is a fixed positive integer m such that, for each $a \in A$, there is an $x \in A$ satisfying $a^{2m}x = a^m$ (in particular, a regular ring may be described as 1-regular); (3) π-*regular* if, for each $a \in A$, there is an $x \in A$ and a positive integer n, depending on a, satisfying $a^{2n}x = a^n$. Regular rings were introduced by von Neumann [4]; m-regular and π-regular rings were first discussed by McCoy [2]. Both of these authors restricted their definitions to rings with identity, but this requirement was subsequently dropped. For some basic properties of regular rings, and for the proof that every regular ring is semi-simple, see [3, pp. 147-149]. (Semi-simplicity is always used in the sense of Jacobson, that is, in the sense that the intersection of the prime maximal ideals is zero.)

If S is a commutative ring with identity such that A admits S as a ring of operators, then A may be embedded in the ring (A; S) with identity defined as follows (see [3, pp. 87-88]): Let $(A; S) = \{(a, s): a \in A, s \in S\}$, and define operations in (A; S) by

$$(a, s) + (b, t) = (a + b, s + t), \quad (a, s) \cdot (b, t) = (ab + sb + ta, st).$$

The identity of (A; S) is the element (0, 1). The subset $A_0 = \{(a, s): s = 0\}$ is easily seen to be an ideal of (A; S) which is isomorphic to the given ring A.

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The symbol n will be reserved for the characteristic of A. We may always choose S to be I_n, the ring of integers modulo (n). (The method used by Stone was the one just described, with $S = I_2$.) In general, there may be no other ring available as a candidate for S. Thus, we shall be concerned with conditions on n which ensure that I_n satisfies the requirements found for S to yield the desired kind of ring with identity. (The first three lemmas are used in that part of the investigation.) However, in cases where I_n is not suitable, another choice is often possible. As a somewhat special example, suppose that A is an algebra over a field F. Then, as will be seen, we may set S = F.

3. Lemmas 1 and 2 are quite simple, but they do not seem to occur in any familiar reference.

LEMMA 1. *If* A *is a commutative ring with no non-zero nilpotent elements, then either* $n = 0$ *or* n *is square-free.*

Proof. Let $n \neq 0$, and suppose $n = p^2q$, where p is prime. Let a be an element of order n. Then $(pqa)^2 = p^2q^2a^2 = n(qa^2) = 0$, but $pqa \neq 0$.

That the converse is false can be seen immediately by considering zero-rings.

LEMMA 2. I_e *is a regular ring if and only if* e *is square-free.*

Proof. Suppose e is square-free, say $e = p_1 \cdots p_k$, where each p_i is prime. Then I_e is isomorphic to the direct sum of the fields $I_{p_1}, \cdots I_{p_k}$, and hence it is regular.

Conversely, if I_e is regular, it contains no non-zero nilpotent elements. By Lemma 1, either $e = 0$ or e is square-free. But it is clear that the ring of integers is not regular.

LEMMA 3. I_e *is an* m*-regular ring for some* m *if and only if* $e \neq 0$.

Proof. The necessity of the condition is obvious; we turn to the sufficiency.

First, it will be shown that the direct sum of a finite number of commutative m_i-regular rings is m-regular for some m. Let $A_1, \cdots, A_k$ be m_1-, $\cdots$, m_k-regular, respectively; let A denote the direct sum of $A_1, \cdots, A_k$; and let $m = m_1 \cdots m_k$. Given $a \in A$, $a = (a_1, \cdots, a_k)$, there exist $x_i \in A_i$ such that $a_i^{2m_i} x_i = a_i^{m_i}$ $(i = 1, \cdots, k)$. Hence

$$a_i^{2m} x_i^{m/m_i} = (a_i^{2m_i} x_i)^{m/m_i} = (a_i^{m_i})^{m/m_i} = a_i^m, \quad (i = 1, \cdots, k).$$

Let $x = (x_1^{m/m_1}, \cdots, x_k^{m/m_k})$. Then $x \in A$, and $a^{2m}x = a^m$.

Next, a finite ring B in which every non-nilpotent element has an inverse is m-regular for some m. If $b \in B$ is non-nilpotent, then $b^2(b^{-1}) = b$. If $b \in B$ is nilpotent with index h, then for any $y \in B$ we have $b^{2h}y = b^h$. Let $h_1, \cdots, h_k$ be the indices of the nilpotent elements of B. Then it follows from equations like those in the preceding paragraph that B is m-regular, with $m = h_1 \cdots h_k$.

If $e = q_1 \cdots q_k$, where $q_1, \cdots, q_k$ are powers of distinct primes, then I_e is the direct sum of $I_{q_1}, \cdots, I_{q_k}$. Now, a trivial modification of the proof that I_p is a field when p is a prime shows that I_{q_i} is a ring in which every non-nilpotent element has an inverse. From the results just established, we conclude that I_e is an m-regular ring, for some m.

The next lemma is essentially known, but is included for convenience of reference.

LEMMA 4. *If* A *is a commutative* m-*regular* (π-*regular*) *ring, and* B *is any ideal of* A, *then* B *is an* m-*regular* (π-*regular*) *ring.*

Proof. If $a \in B$, there is an $x \in A$ (and an integer m) such that $a^{2m}x = a^m$. Set $y = a^m x^2$. Then $y \in B$, and $a^{2m}y = a^m$.

The final lemma is the main result needed to establish the theorems which follow.

LEMMA 5. *Let* A *be a commutative, semi-simple,* π-*regular ring. The validity of the relations* $a^{2m_1}y = a^{m_1}$ *in* A *and* $s^{2m_2}t = s^{m_2}$ *in* S *implies that, for some* $x \in A$, *the relation*

$$(a, s)^{2m_1m_2}(x, t^{m_1}) = (a, s)^{m_1m_2}$$

holds in (A; S).

Proof. It must be shown that there is an $x \in A$ such that $\beta x + t^{m_1}\beta + s^{2m}x = \alpha$, where $m = m_1m_2$,

$$\alpha = \sum_{k=0}^{m-1} \binom{m}{k} s^k a^{m-k}, \quad \text{and } \beta = \sum_{k=0}^{2m-1} \binom{2m}{k} s^k a^{2m-k};$$

we write this in the form

$$\beta x + s^{2m}x = \alpha - t^{m_1}\beta. \tag{1}$$

Since A is semi-simple, it suffices to solve the corresponding congruence modulo M, for every prime maximal ideal M. Define $c = a + sa^m y^{m_2}$. Because $a^m y^{m_2}$ is idempotent, we have

$$c^m = \alpha a^m y^{m_2} + s^m a^m y^{m_2}, \quad c^{2m} = \beta a^m y^{m_2} + s^{2m} a^m y^{m_2};$$

and using the equation $s^{2m}t^{m_1} = s^m$, we obtain

$$c^m - t^{m_1}c^{2m} = (\alpha - t^{m_1}\beta)a^m y^{m_2}. \tag{2}$$

By π-regularity, there is a $z \in A$ and an integer p satisfying $c^{2p}z = c^p$. Let k be an integer such that $pk > 2m$. Since c^pz is idempotent,

$$c^{2m}(c^{pk-2m}z^k) = c^{pk}z^k = (c^pz)^k = c^pz. \tag{3}$$

We set $x = c^{pk-2m}z^k(c^m - t^{m_1}c^{2m})$. Then $x = a^m y^{m_2}x$; for if $a \in M$, then $c \in M$, whence $x \in M$; while if $a \notin M$, then $a^m \notin M$, and $a^m y^{m_2}$ is the identity modulo M. Hence,

$$\beta x + s^{2m}x = \beta a^m y^{m_2}x + s^{2m}a^m y^{m_2}x = c^{2m}x,$$

and (1) becomes $c^{2m}x = \alpha - t^{m_1}\beta$, that is, from (2), $c^{2m}xa^m y^{m_2} = c^m - t^{m_1}c^{2m}$, or $c^{2m}x = c^m - t^{m_1}c^{2m}$. Thus, in view of (3), to show that x is the required solution we need only verify that

$$c^p z(c^m - t^{m_1} c^{2m}) = c^m - t^{m_1} c^{2m}.$$

But if $c \in M$, this is trivial; while if $c \notin M$ (whence $c^p \notin M$), then $c^p z$ is the identity modulo M.

THEOREM 1. *Let* A *be a commutative ring. The ring* (A; S) *is regular if and only if* A *and* S *are regular. In particular,* $(A; I_n)$ *is regular if and only if* A *is regular and* $n \neq 0$.

Proof. If (A; S) is regular, then for any $(a, s) \in (A; S)$, there is an $(x, t) \in (A; S)$ such that $(a, s)^2(x, t) = (a, s)$, and thus $s^2t = s$. Hence, S is regular. Since A is isomorphic to the ideal A_0 of (A; S), Lemma 4 (with $m = 1$) shows that A is regular.

For the converse, apply Lemma 5 with $m_1 = m_2 = 1$.

Now suppose that $(A; I_n)$ is regular. Then I_n is regular, so $n \neq 0$.

Conversely, let A be regular and $n \neq 0$. Since A has no non-zero nilpotent elements, it follows from Lemmas 1 and 2 that I_n is regular. Hence $(A; I_n)$ is regular.

THEOREM 2. *If* A *is a commutative, semi-simple,* m_1*-regular ring, and* S *is* m_2*-regular (in particular, if* $S = I_n$ *with* $n \neq 0$*), then* (A; S) *is* m_1m_2*-regular. If* (A; S) *is* m_3*-regular, then* A *and* S *are* m_3*-regular. (In particular, if* $(A; I_n)$ *is* m_3*-regular, then* $n \neq 0$.)

Proof. The first statement follows from Lemma 5, with m_1 and m_2 fixed, while the verification of the second statement is analogous to the first paragraph of the proof of Theorem 1. The parenthetical remarks may be obtained by the use of Lemma 3.

THEOREM 3. *If* A *is a commutative, semi-simple,* π*-regular ring, and* S *is* π*-regular (in particular, if* $S = I_n$ *with* $n \neq 0$*), then* (A; S) *is* π*-regular. If* (A; S) *is* π*-regular, then* A *and* S *are* π*-regular. (In particular, if* $(A; I_n)$ *is* π*-regular, then* $n \neq 0$.)

The proof is similar to that of Theorem 2.

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