NOTE ON AN ENUMERATION THEOREM OF DAVIS AND SLEPIAN

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1. INTRODUCTION

In recent papers, Davis [1] and Slepian [6, 7] independently obtained an elegant combinatorial lemma which proved to be extremely useful for solving certain enumeration problems. Although Davis ([1], equation (1), or briefly [1, (1)]), gives the number of nonisomorphic structures of m-adic relations on n elements, and Slepian [7, (2)] gives the number of types of Boolean functions of n variables, it is clear that their methods are combinatorially identical. The lemma was also found useful by Gilbert [2] in enumerating types of periodic sequences.

Applying this lemma, Slepian [7, (3)] and Davis [1, Theorems 2, 3, 4, 5] obtained formulas for various enumeration problems which are all of the following general form: the desired number of configurations is equal to the reciprocal of the order of the appropriate permutation group, multiplied by a sum (taken over the types of permutations in this group) of terms each of which is the product of the number of group elements of the corresponding type by some power of the number of figures which serve as building blocks for the configurations; in each of these powers, the exponent is equal to the number of cycles in the permutation of this type.

The objects of this note are to state the enumeration formula of Davis and Slepian explicitly, and to show that it may readily be derived from Pólya's enumeration theorem, the Hauptsatz of [5]. We also illustrate the use of the Davis-Slepian Theorem by enumerating the types of Boolean functions of two variables, following Slepian [7] and Pólya [4], and by finding the number of (nonisomorphic) linear graphs on four vertices, following Davis [1], Slepian [6], and Harary [3]. We conclude by stating a formula which combines Pólya's theorem with the Davis-Slepian Theorem, and which is applicable to enumeration problems that involve configurations containing figures whose "content" has more than one dimension.

2. TERMINOLOGY

First we develop a precise statement of the Davis-Slepian Theorem. Let Φ be a set of elements, called *figures*. A *configuration of length* s is a sequence of s figures. Let Γ be a permutation group of degree s and order h. We say that two configurations are *Γ-equivalent* if there exists a permutation in Γ which sends one onto the other. Let b be the number of figures (that is, elements) of Φ, and let B be the number of Γ-equivalence classes of configurations (in other words, the number of Γ-inequivalent configurations). A permutation in Γ, written as a product of disjoint cycles, is of *type* $(j) = (j_1, j_2, \cdots, j_s)$ if it has j_k cycles of length k for $k = 1, \cdots, s$. Let $h_{(j)}$ be the number of elements of Γ of type (j). We now have the notation necessary for stating the formula of Davis and Slepian,

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(1) $$B = \frac{1}{h} \sum_{(j)} h_{(j)} b^{\Sigma_{k=1}^{s} j_k}$$

In order to be able to state Pólya's theorem symbolically, we need the following additional concepts. Let each figure have associated with it an n-tuple of nonnegative integers $(m_1, m_2, \cdots, m_n)$. This n-tuple is called the *content of the figure*, and n is the *dimension* of the content. The *content of a configuration* is the vector sum of the contents of the figures in it. For convenience, we shall state Pólya's theorem for contents of dimension 2. Let b_{mn} be the number of figures of content (m, n), and let

(2) $$b(x, y) = \sum_{m,n=0}^{\infty} b_{mn} x^m y^n$$

be the *figure-counting series*. Similarly we denote the number of Γ-inequivalent configurations of content (m, n) by B_{mn}, and write

(3) $$B(x, y) = \sum_{m,n=0}^{\infty} B_{mn} x^m y^n$$

for the *configuration-counting series*. Pólya's Theorem expresses B(x, y) in terms of b(x, y) and the permutation group Γ.

Let $f_1, f_2, \cdots, f_s$ be s indeterminates. Then the *cycle index* of Γ is defined by

(4) $$Z(\Gamma) = \frac{1}{h} \sum_{(j)} h_{(j)} f_1^{j_1} \cdots f_s^{j_s}.$$

For any power series p(x, y), let $Z(\Gamma, p(x, y))$ be the series obtained from $Z(\Gamma)$ on replacing each indeterminate f_k by $p(x^k, y^k)$.

Pólya's Theorem.

(5) $$B(x, y) = Z(\Gamma, b(x, y)).$$

3. DERIVATION OF THE DAVIS-SLEPIAN THEOREM FROM PÓLYA'S THEOREM

For figures with content of dimension one, equation (5) becomes

(5') $$B(x) = Z(\Gamma, b(x)),$$

and (4) transforms this to

(6) $$B(x) = \frac{1}{h} \sum_{(j)} h_{(j)} b^{j_1}(x) b^{j_2}(x^2) \cdots b^{j_s}(x^s).$$

But the total number b of figures regardless of content, and the total number B of Γ-inequivalent configurations, are given by

$$b = b(1) \quad \text{and} \quad B = B(1). \tag{7}$$

Hence

$$B = B(1) = \frac{1}{h}\sum_{(j)} h_{(j)} b^{j_1}(1) b^{j_2}(1) \cdots b^{j_s}(1)$$

$$= \frac{1}{h}\sum_{(j)} h_{(j)} b^{\Sigma_{k=1}^{s} j_k};$$

this is the formula (1). It is clear from (1) that to find B in closed form one needs to know the number b of figures, ord $\Gamma = h$, and the number $h_{(j)}$ of permutations in Γ of each type (j); finally one must have a formula for $\Sigma_{k=1}^{s} j_k$. It is this last formula which is usually complicated from the combinatorial point of view.

Thus Pólya's theorem is useful in finding the number of (Γ-inequivalent) configurations *with a given content*, while the Davis-Slepian Theorem gives more conveniently the number of configurations *without regard to content.* It is interesting to note that both in [6] and [7] Slepian succeeds in supplying additional combinatorial steps which permit the derivation of (5) from (1), for his enumeration problems. These steps are the same as the last few steps in Pólya's proof of his Hauptsatz.

4. EXAMPLES

We illustrate equation (1) first by computing the total number of (nonisomorphic) graphs with 4 vertices. To do this we make use of the example given in [1] which shows the derivation of the counting polynomial for graphs of 4 vertices. Each configuration in this context is a graph of 4 vertices, and is of length 6. The 6 places in a configuration are the 6 unordered pairs taken from the 4 vertices. The number b of figures is 2, since any two distinct vertices are either adjacent or nonadjacent in a graph. The configuration group Γ is the pair group of the symmetric group of degree 4, with cycle index

$$Z(\Gamma) = \frac{1}{24}(f_1^6 + 9f_1^2f_2^2 + 8f_3^2 + 6f_2f_4).$$

Therefore an application of (1) yields for g_4, the desired number of graphs of 4 vertices,

$$g_4 = \frac{1}{24}(2^6 + 9\cdot 2^4 + 8\cdot 2^2 + 6\cdot 2^2) = 11.$$

This is the same computation as given by Davis [1] with the notation irs (4) (the number of nonisomorphic irreflexive symmetric dyadic relations on 4 elements) for g_4, as well as the number of types of linear graphs of 4 vertices given by Slepian [6].

The enumeration of types of Boolean functions of n variables was derived by Pólya [4] for small values of n, and by Slepian [7] for arbitrary n. For Boolean functions $f(x_1, x_2)$ of two variables, the canonical form (with x_i' denoting the negation of x_i) is

$$f(x_1, x_2) = \varepsilon_1 x_1 x_2 + \varepsilon_2 x_1 x_2' + \varepsilon_3 x_1' x_2 + \varepsilon_4 x_1' x_2',$$

where the ε_j are either 0 or 1. Thus a Boolean function of two variables is a configuration of length 4, since there are 4 terms in this expansion. The number of figures is 2, since there are 2 admissible values for each coefficient ε_j. Two Boolean functions of n variables are of the same type if one can be obtained from the other by some permutation of the n variables, followed by the negation of some subset of the variables. Hence the configuration group is the permutation group of the symmetries of the n-cube. For n = 2, this is the dihedral group D_4 of degree 4, whose cycle index is

$$Z(D_4) = \frac{1}{8}(f_1^4 + 2f_1^2f_2 + 3f_2^2 + 2f_4).$$

Thus, by applying (1), we find that the number of types of Boolean functions of two variables is

$$B_2 = \frac{1}{8}(2^4 + 2 \cdot 2^3 + 3 \cdot 2^2 + 2 \cdot 2) = 6.$$

These 6 types of Boolean functions are exhibited in detail by Pólya [4].

Let us consider configurations whose figures have content of dimension greater than 1, and illustrate with dimension 2. Then the following specializations of (5) enable us to enumerate the configurations in terms of

(a) the content of the first dimension only,
(b) the content of the second dimension only, or
(c) without regard to content:

$$B(x, 1) = Z(\Gamma, b(x, 1)), \tag{8a}$$

$$B(1, y) = Z(\Gamma, b(1, y)), \tag{8b}$$

$$B(1, 1) = Z(\Gamma, b(1, 1)). \tag{8c}$$

Obviously (8c) is the same formula as (1). Such considerations may be applicable to the intriguing unsolved problem of enumerating the nonisomorphic abstract simplicial complexes of dimension n.

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