A NOTE ON POWER SERIES AND AREA

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Let C denote the unit circle $|z| = 1$, and D the open unit disk $|z| < 1$ in the complex plane. We shall find it convenient to refer to the intersection of D with a neighborhood of a point $e^{i\theta} \in C$ as a "neighborhood of $e^{i\theta}$." If $f(z)$ is a holomorphic function in D, we call $e^{i\theta}$ a *strong point* of $f(z)$ provided that every neighborhood of $e^{i\theta}$ is mapped by $f(z)$ onto a Riemann configuration of infinite area; otherwise we call $e^{i\theta}$ a *weak point* of $f(z)$. Every strong point of $f(z)$ is obviously also a singular point of this function. The converse, however, is not true, for if $f(z)$ is schlicht and maps D onto a region bounded by a Jordan curve possessing no analytic subarc, then every $e^{i\theta}$ is a singular point as well as a weak point of $f(z)$.

We are going to modify a result (and its proof) due to Ryll-Nardzewski and Steinhaus [5; 1, p. 102] to show that as a rule (in a certain sense) $f(z)$ has every $e^{i\theta}$ for a strong point. We are indebted to W. Seidel for some helpful suggestions.

THEOREM. *For every* x *in some Banach space* X, *let* $f(x, z)$ *be a holomorphic function of* $z \in D$, *and for every* $z \in D$, *let* $f(x, z)$ *be a linear functional of* $x \in X$. *Then there exists an open set* $G \subset C$, *and a set* $Q \subset X$, *of type* F_σ *and of first category, such that, for every* $x \in X$, *every* $e^{i\theta} \in G$ *is a weak point of* $f(x, z)$, *and, for every* $x \in X - Q$ *(a residual subset of* X*), every* $e^{i\theta} \in C - G$ *is a strong point of* $f(x, z)$.

If, further, to every $e^{i\theta} \in C$ *there corresponds an* $x_\theta \in X$ *such that* $e^{i\theta}$ *is a strong point of* $f(x_\theta, z)$, *then, for every* $x \in X - Q$, *every* $e^{i\theta} \in C$ *is a strong point of* $f(x, z)$.

Before proving the theorem, we consider the following example. Let

$$f(x, z) = \sum_{k=0}^{\infty} a_k z^k, \quad x = \{a_k\}, \quad a_k \text{ complex} \quad (k = 0, 1, 2, \cdots),$$

and take X to be the Banach space which consists of all bounded sequences x, with $\|\{a_k\}\| = \sup_k |a_k|$ (this is the space X_2 of Ryll-Nardzewski and Steinhaus [5; 1, p. 104]). According to Lusin [3; 2, p. 69], there exists a power series $\sum b_k z^k$, with $\lim_{k\to\infty} b_k = 0$, which diverges at every point of C. If we set $\beta = \{b_k\}$ $(k = 0, 1, 2, \cdots)$, then $\beta \in X$, and it follows from a result of Zygmund [7] that every point $e^{i\theta}$ is a strong point of $f(\beta, z)$. Consequently, according to our theorem, there exists a residual set $R \subset X$ such that, for every $x \in R$, $f(x, z)$ maps every neighborhood of every point of C onto a Riemann configuration of infinite area.

Proof of the theorem. By a *rational arc* we mean an open subarc of C whose end points have principal amplitudes that are rational numbers. We call a rational arc A a *weak arc* provided that, for every $x \in X$, every $e^{i\theta} \in A$ is a weak point of $f(x, z)$. Denote the set of all weak arcs by W, let G be the union of all weak arcs, and set $H = C - G$. Then obviously, for every $x \in X$, every $e^{i\theta} \in G$ is a weak point of $f(x, z)$.

For every natural number n and every rational arc A, we denote the region

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$$1 - 1/n < |z| < 1, \quad z/|z| \in A$$

by $D_n(A)$, and we define $E_n(A)$ to be the set of all $x \in X$ such that $f(x, z)$ maps $D_n(A)$ onto a Riemann configuration of area not greater than n. Then it is evident that

$$E_1(A) \subset E_2(A) \subset \cdots \subset E_n(A) \subset \cdots . \tag{1}$$

The functions $f(x, z)$ $(x \in E_n(A))$ form a normal family in $D_n(A)$ [4]. Now $E_n(A)$ *is closed in* X. For suppose that $x_k \in E_n(A)$ $(k = 1, 2, 3, \cdots)$ and $\lim_{k\to\infty} x_k = x^*$. Then, for every $z \in D_n(A)$, since $f(x, z)$ is a linear functional of x, we have

$$\lim_{k\to\infty} f(x_k, z) = f(x^*, z).$$

Because of the normality of the family $\{f(x, z)\}$ $(x \in E_n(A))$, there exists a subsequence $\{f(x_{k_j}, z)\}$ $(j = 1, 2, 3, \cdots)$ of $\{f(x_k, z)\}$ which converges uniformly to $f(x^*, z)$ in every closed subregion of $D_n(A)$, and then the sequence $\{f'(x_{k_j}, z)\}$ $(j = 1, 2, 3, \cdots)$ of the corresponding derivatives with respect to z converges to $f'(x^*, z)$ in $D_n(A)$. Consequently, if $d\sigma$ denotes an element of area in the z-plane, an application of Fatou's lemma [6, p. 29] yields

$$\iint_{D_n(A)} |f'(x^*, z)|^2 d\sigma = \iint_{D_n(A)} \lim_{j\to\infty} |f'(x_{k_j}, z)|^2 d\sigma \leq \underline{\lim}_{j\to\infty} \iint_{D_n(A)} |f'(x_{k_j}, z)|^2 d\sigma \leq n,$$

and hence x^* also belongs to $E_n(A)$.

If A is a rational arc and $A \nsubseteq W$, then $\bigcup_{n=1}^{\infty} E_n(A)$ *is nowhere dense in* X. For suppose that ξ is an interior point of $\bigcup_{n=1}^{\infty} E_n(A)$. Then, for every $x \in X$, there exists a real number $t > 0$ such that $\xi + tx \in \bigcup_{n=1}^{\infty} E_n(A)$. Since $f(x, z)$ is a linear functional of x,

$$f(x, z) = \frac{1}{t}\,[f(\xi + tx, z) - f(\xi, z)] .$$

There exist natural numbers n_1, n_2 such that $\xi \in E_{n_1}(A)$ and $\xi + tx \in E_{n_2}(A)$, and if we set $m = \max(n_1, n_2)$, we have, because of (1), $\xi \in E_m(A)$ and $\xi + tx \in E_m(A)$. Now, if we bear in mind Schwarz's inequality, we obtain

$$\iint_{D_m(A)} |f'(x, z)|^2 d\sigma = \frac{1}{t^2} \iint_{D_m(A)} |f'(\xi + tx, z) - f'(\xi, z)|^2 d\sigma$$

$$\leq \frac{1}{t^2}\left\{ \iint_{D_m(A)} |f'(\xi + tx, z)|^2 d\sigma + \iint_{D_m(A)} |f'(\xi, z)|^2 d\sigma + 2 \iint_{D_m(A)} |f'(\xi + tx, z)| \cdot |f'(\xi, z)|\, d\sigma \right\}$$

$$\leq \frac{2}{t^2}\left\{ m + \left(\iint_{D_m(A)} |f'(\xi + tx, z)|^2 d\sigma \cdot \iint_{D_m(A)} |f'(\xi, z)|^2 d\sigma \right)^{\frac{1}{2}} \right\} \leq \frac{4m}{t^2}.$$

Hence, if N is a sufficiently large natural number, $x \in E_N(A) \subset \bigcup_{n=1}^{\infty} E_n(A)$. This implies, however, that A is a weak arc, which contradicts our assumption that $A \notin W$.

There are only enumerably many rational arcs, and therefore, if we set

$$(2) \qquad Q = \bigcup_{A \notin W} \bigcup_{n=1}^{\infty} E_n(A) \qquad \text{(where A denotes a rational arc),}$$

then Q is a set of type F_σ and of first category; let $R = X - Q$.

Suppose that $x \in R$, $e^{i\theta} \in H$, and $e^{i\theta}$ is a weak point of $f(x, z)$. Then $f(x, z)$ maps some neighborhood of $e^{i\theta}$ onto a Riemann configuration of finite area. This implies that, for some sufficiently small rational arc A containing $e^{i\theta}$, and some sufficiently large natural number n, $x \in E_n(A)$. Since $x \notin Q$, it follows from (2) that $A \in W$, and hence $e^{i\theta} \in G$, which contradicts our assumption that $e^{i\theta} \in H$. This means that, for every $x \in R$, every $e^{i\theta} \in H$ is a strong point of $f(x, z)$.

If X is such that to every $e^{i\theta} \in C$ there corresponds an $x_\theta \in X$ with the property that $e^{i\theta}$ is a strong point of $f(x_\theta, z)$, then G is empty and $H = C$, so that, for every $x \in R$, every $e^{i\theta} \in C$ is a strong point of $f(x, z)$.

This completes the proof of the theorem.

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