SOME REMARKS ON SET THEORY IV

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1. SOME PROBLEMS OF SIERPIŃSKI. Sierpiński [6], [7, pp. 9-11] proved that the continuum hypothesis is equivalent with the existence of a decomposition of the plane into two sets S_1 and S_2 such that S_1 is intersected by every horizontal line (and S_2 by every vertical line) in at most a denumerable set. We begin with a generalization of this result.

THEOREM 1. *Assume that* $2^{\aleph_0} = \aleph_1$. *Decompose the set of all lines in the plane into two arbitrary disjoint sets* L_1 and L_2. *Then there exists a decomposition of the plane into two sets* S_1 *and* S_2 *such that each line of* L_i *intersects* S_i $(i = 1, 2)$ *in at most a denumerable set.*

This theorem clearly strengthens one part of Sierpiński's result. To prove the theorem, let $\{l_\alpha\}$ $(\alpha < \Omega_1)$ be a well-ordering of the lines in the plane, and let l_1 belong to L_i. We begin the construction of the sets S_1 and S_2 by assigning all points of l_1 to S_{3-i}. Suppose that for $\beta < \alpha$ the points of the lines l_β have been divided between S_1 and S_2, and that l_α belongs to L_i. Then we assign to S_{3-i} all points of l_α which lie on none of the lines l_β $(\beta < \alpha)$. The sets S_1 and S_2, thus defined by transfinite induction, possess the required properties, since each ordinal less than Ω_1 is denumerable.

If we do not appeal to the continuum hypothesis, our proof gives a decomposition of the plane into two sets S_i $(i = 1, 2)$ such that each line of L_i intersects S_i in a set of power less than $2^{\aleph_0}$. (Compare Sierpiński's remark immediately after Theorem 4 on page 6 of [8].)

The other half of Sierpiński's theorem can also be strengthened. To this end, we want to find necessary and sufficient conditions on two disjoint sets of lines L_1 and L_2 such that the existence of a decomposition of the plane into two sets S_1 and S_2, with every line of L_i intersecting S_i $(i = 1, 2)$ in at most a denumerable set, implies the continuum hypothesis. Such conditions may be stated as follows: Both L_1 and L_2 must contain nondenumerably many lines, and one of them, say L_1, must contain $2^{\aleph_0}$ lines; moreover, there must not exist a point p such that all but $\aleph_1$ lines of L_1 and all but $\aleph_0$ lines of L_2 pass through p.

We suppress the proof, since it is somewhat lengthy and contains no ideas which are not involved in Sierpiński's method [6, p. 2], [7, pp. 10, 11]. Just to give a hint to the reader, we remark that in the proof we distinguish two cases: in Case I, if T is any set of power $\aleph_1$, some line of L_1 does not meet T; in Case II, this condition is not satisfied.

Various problems arise in connection with Theorem 1. Sierpiński [9] proved that the continuum hypothesis is equivalent with the following statement: *Three-dimensional space* E^3 *can be decomposed into three sets* S_i $(i = 1, 2, 3)$ *such that each line parallel to one of the axes* OX_i $(i = 1, 2, 3)$ *intersects* S_i *in a finite set.* This suggests several questions:

a) Distribute the lines in E^3 into three arbitrary sets L_i $(i = 1, 2, 3)$. Does there exist a decomposition of E^3 into three sets S_i such that the intersection of each line of L_i with the corresponding set S_i is finite?

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b) Does there exist a decomposition of the plane into three sets S_i $(i = 1, 2, 3)$ such that each horizontal (vertical, oblique) line intersects S_1 (S_2, S_3) in a finite set?

c) Does there exist a set of three directions d_i $(i = 1, 2, 3)$ in the plane, together with a decomposition of the plane into three corresponding sets S_i, such that every line with the direction d_i intersects S_i in a finite set?

The three questions deal with progressively weaker conjectures; the last is due to Sierpiński. I do not know the answer to any of them.

Sierpiński's theorem at the beginning of this section can clearly be formulated in the following more general form. Let T be a set of power m. By T^2 we denote the set of all pairs (a, b) with $a \in T$ and $b \in T$. A horizontal line $l_{(a)}$ is defined as a set $\{(a, x)\}$, where $a \in T$ and x runs through all elements of T; and a vertical line $l^{(b)}$ is defined as a set $\{(x, b)\}$, where $b \in T$ and x runs through all elements of T. Then T^2 can be decomposed into two sets S_1 and S_2 such that S_1 is intersected by every horizontal line (and S_2 by every vertical line) in a set of power less than m. Now we prove the following result.

THEOREM 2. *Let* n *be a cardinal number less than* m, *and let* $T^2 = S_1 \cup S_2$, *where* S_1 *is intersected by every horizontal line in a set of power less than* n. *Then there exists a vertical line which intersects* S_2 *in a set of power* m.

Theorem 2 is essentially due to Sierpiński [8, p. 6], though I am not sure that he ever stated it explicitly; for the sake of completeness I give a proof. Denote by S_a the set of those x for which the point (a, x) lies in $l_a \cap S_1$. By assumption, $\overline{\overline{S_a}} < n < m$. Thus by the lemma on page 55 of [2] there exists a subset T_1 of T, of power m, such that the union of all sets S_a $(a \in T_1)$ is a proper subset of T; in other words, such that there exists an element b of T not contained in this union. But then clearly all the points (x, b) with x in T_1 are in S_2; that is, the vertical line $l^{(b)}$ meets S_2 in a set of power m, as stated.

If we use the generalized continuum hypothesis, we can prove the following stronger result.

THEOREM 3. *Let* n *be any cardinal number less than* m. *Let* $T^2 = S_1 \cup S_2$, *where* S_1 *is intersected by every horizontal line in a set of power less than* n. *Then there exist two subsets* T_1 *and* T_2 *of* T, *each of power* m, *such that all points* (a, b) *with* a *in* T_1 *and* b *in* T_2 *belong to* S_2.

By the lemma[1] on page 56 of [2], there exist two sets $T_1 \subset T$ and $T_2 \subset T$, of power m, such that the set

$$T_2 \cap \Big(\bigcup_{a \in T_1} S_a \Big)$$

is empty. Thus all points (a, b) with a in T_1 and b in T_2 are in S_2, as stated.

Sierpiński [8] proved that if $2^{\aleph_0} = \aleph_1$, the plane is the union of countably many curves (here the word 'curve' is used to denote a point set C for which there exists a direction such that every line in this direction intersects the set C in at most one point). Earlier, Mazurkiewicz [5] had proved that the plane is not the union of

1. G. Fodor has recently proved this lemma without using the continuum hypothesis; the proof will appear in Acta Litt. Sci. Szeged.

finitely many curves, and Sierpiński [8, p. 8] raised the question whether the continuum hypothesis is equivalent with the statement that the plane is the union of countably many curves.

The following theorem proves a conjecture of L. Patai, regarding the decomposition of the plane into countably many curves; I read the conjecture in a note book of Patai, after his death.

THEOREM 4. *Assume that* $2^{\aleph_0} = \aleph_1$. *Let there be given a set of directions* d_n $(n = 1, 2, \cdots)$ *in the plane such that, for every direction* d *in the plane, infinitely many of the* d_n *are different from* d. *Then the plane is the union of countably many curves* C_n *with the property that, for each* n, C_n *intersects every line parallel to* d_n *in at most one point.*

By hypothesis the set $\{d_n\}$ can be split into two infinite sets $\{d_i'\}$ and $\{d_j''\}$ $(i, j = 1, 2, \cdots)$ such that no d_i' is parallel to any d_j''. By Theorem 1 the plane can be split into two sets S_1 and S_2 such that each line parallel to one of the d_i' $(i = 1, 2, \cdots)$ intersects S_1 in at most a countable set, and each line parallel to one of the d_j'' $(j = 1, 2, \cdots)$ intersects S_2 in at most a countable set.

We will say that two points u and v of S_1 belong to the same class provided there exists a finite set of points $\{t_1, t_2 \cdots, t_k\}$, with $t_1 = u$ and $t_k = v$, such that each line (t_r, t_{r+1}) $(r = 1, 2, \cdots, k - 1)$ is parallel to one of the directions d_i'. The set S_1 is thus partitioned into disjoint classes B_α $(1 \leq \alpha < \Omega_1)$; and since each line parallel to one of the d_i' intersects S_1 in at most a denumerable set, each class B_α is at most denumerable. We denote its points by $t_{\alpha n}$ $(n = 1, 2, \cdots)$.

For $n = 1, 2, \cdots$, let C_n' denote the set $\{t_{\alpha n}\}$ $(1 \leq \alpha < \Omega_1)$. Any two distinct points $t_{\alpha_1 n}$ and $t_{\alpha_2 n}$ in C_n' belong to two different classes; therefore the line joining them is not parallel to any of the d_i' $(i = 1, 2, \cdots)$. It follows that C_n' is a curve which meets each line parallel to one of the d_i' in at most one point.

Similarly, the set S_2 can be decomposed into countably many curves C_n'', each of which meets each line parallel to one of the d_j'' in at most one point. This completes the proof of Theorem 4.

The condition that for every direction d there exist infinitely many d_n not parallel to d can not be omitted from the hypothesis of Theorem 4. In fact, a simple modification of the proof of Mazurkiewicz [5] yields the following result: *The plane is not the union of a finite number of curves* $C_1, C_2, \cdots, C_n$ *and a set* S *such that every line parallel to a certain direction* d *intersects* S *in fewer than* $2^{\aleph_0}$ *points.* We do not give the proof, since it is very similar to that of Mazurkiewicz.

The following question is now appropriate. Let the lines in the plane be divided into countably many disjoint classes L_n $(n = 1, 2, \cdots)$. Can one then split the plane into countably many sets S_n $(n = 1, 2, \cdots)$ such that each line of L_n intersects S_n in at most one point? I do not know the answer to this question. If it is in the affirmative, it clearly sharpens Theorem 4.

2. ON ADDITIVE NUMBER THEORY. Lorentz [3] recently proved the following conjecture of E. G. Straus and myself: *Let* $\{a_i\}_1^\infty$ *be any increasing sequence of integers. Then there exists an increasing sequence* $\{b_j\}$, *of density* 0, *such that every sufficiently large integer is of the form* $a_i + b_j$. The following theorem shows that no analogous result holds for real numbers. In the statement of the theorem, the sum A + B of two sets of numbers denotes the set of all numbers a + b with a in A and b in B.

THEOREM 5. *Assume the continuum hypothesis, and suppose that* $\{S_\alpha\}$ $(1 \leq \alpha < \Omega_1)$ *is a family of* $\aleph_1$ *sets of real numbers, with the property that the set of all real numbers is not the union of countably many sets of the form* $a_k + S_{\alpha_k}$ $(k = 1, 2, \cdots)$. *Then there exists a set* A *of power* $\aleph_1$ *such that none of the sets* $A + S_\alpha$ $(1 \leq \alpha < \Omega_1)$ *is the set of all real numbers.*

Before proving Theorem 5, we note that every family of $\aleph_1$ sets of measure zero (in particular, the family of sets of measure zero and type G_δ) satisfies the hypothesis of the theorem; the same is true of every family of $\aleph_1$ sets of first category. I do not know whether the conclusion of the theorem holds whenever $\{S_\alpha\}$ consists of $\aleph_1$ sets of measure zero and $\aleph_1$ sets of first category (since the union of two such sets can be the set of all real numbers, the hypothesis of Theorem 5 is here not satisfied).

We will now prove our theorem by constructing, by transfinite induction, two sequences of real numbers y_α and z_α $(1 \leq \alpha < \Omega_1)$; the union of the elements of $\{y_\alpha\}$ will serve as the set A; and each set $A + S_\alpha$ will fail to contain the number z_α.

Suppose that, for $\gamma < \beta$, the numbers y_γ and z_γ have been chosen in such a way that, for every $\alpha < \beta$, each of the sets $\{y_\gamma\} + S_\alpha$ $(\gamma < \beta)$ fails to contain z_α. By the hypothesis concerning the family $\{S_\alpha\}$, the union $\cup(S_\beta + y_\gamma)$ $(\gamma < \beta)$ is not the set of real numbers; therefore we can choose a number z_β in the complement of this union. Also, the union $\cup[S_\alpha + (-z_\alpha)]$ $(\alpha \leq \beta)$ is not the set of real numbers (again, by the hypothesis in the theorem), and therefore we can chose $-y_\beta$ in the complement of this union.

Thus the set $A = \{y_\alpha\}$ $(1 \leq \alpha < \Omega_1)$ is defined by transfinite induction. Each z_α is a member of none of the sets $S_\alpha + \{y_\gamma\}$ $(\gamma < \alpha)$, and of none of the sets $S_\alpha + \{y_\beta\}$ $(\beta \geq \alpha)$, and therefore the proof is complete.

3. ON A PROBLEM OF MARCZEWSKI. A countably additive measure μ defined in a space M is said to be *separable* if there exists in M a sequence of measurable sets V_n with the property that, for each measurable set E in M and every $\eta > 0$, there exists an index n such that

$$\mu(V_n - E) + \mu(E - V_n) < \eta.$$

A family F of sets is said to have the property (k) if each nondenumerable collection $\{S_\alpha\}$ of sets in F contains a nondenumerable subcollection such that each pair of sets in this subcollection has a nonempty intersection.

Marczewski [4, pp. 129, 130] proved that if μ is a separable measure, then the family of sets of positive measure has the property (k), and he asked whether the theorem remains true when the condition of separability of μ is dropped. I will now show that, under a certain condition on M and μ, the answer is in the affirmative.

THEOREM 6. *Let* M *be a set, and* μ *a measure defined on some subsets of* M; *and let* M *be the union of countably many sets of finite measure. Then the family of sets of positive measure has the property* (k).

It will clearly be sufficient to prove that if $\{S_\alpha\}$ $(1 \leq \alpha < \Omega_1)$ is any family of distinct sets of positive measure, there exists a subfamily $\{S_{\alpha_k}\}$ $(1 \leq k < \Omega_1)$ such that $\mu(S_{\alpha_{k'}} - S_{\alpha_{k''}}) > 0$ for each pair of ordinals k' and k" less than Ω_1.

Let $M = \cup_1^\infty T_n$, where each T_n is a set of finite positive measure, with $T_n \neq T_m$ for $n \neq m$. For at least one n there exist $\aleph_1$ sets S_α whose intersections with T_n have positive measure; in fact, there exists a collection of $\aleph_1$ distinct sets S_α for which $\mu(T_n \cap S_\alpha) > \varepsilon$, where ε is some positive constant. We denote these sets by S_α^*, and with the collection $\{S_\alpha^*\}$ we associate an abstract graph G as follows: with each S_α^* we associate an (abstract) vertex p_α; two vertices $p_{\alpha'}$ and $p_{\alpha''}$ are connected by an edge if and only if $\mu(S_{\alpha'}^* \cap S_{\alpha''}^*) > 0$. A simple argument shows that among any r vertices p_{α_i} $(i = 1, 2, \cdots, r)$ with

$$r > \mu(T_n)/\varepsilon \quad \left(\mu(T_n \cap S_\alpha^*) > \varepsilon\right),$$

at least two are connected by an edge. By a result of Dushnik and Miller [1, Theorem 5.22], there exists a set of $\aleph_1$ vertices p_{α_k} $(1 \leq k < \Omega_1)$ any two of which are connected by an edge; in other words, for any pair k' and k" of ordinals less than Ω_1, $\mu(S_{\alpha_{k'}} \cap S_{\alpha_{k''}}) > 0$. This proves the theorem.

One final problem: Does there exist a family of $2^{\aleph_0}$ sets of real numbers, each of positive measure, such that the intersection of any $\aleph_1$ of them is empty? If $2^{\aleph_0} = \aleph_1$, it is quite easy to show this.

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