ON THE SCHWARZ REFLECTION PRINCIPLE

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Let $f(z)$ be meromorphic in the unit circle K: $|z| < 1$, and let the modulus $|f(re^{i\theta})|$, $z = re^{i\theta}$, have radial limit 1 for almost all $e^{i\theta}$ belonging to some arc A: $0 \leq \theta_1 < \theta < \theta_2 \leq 2\pi$ of $|z| = 1$. We are interested in conditions under which $f(z)$ may be continued analytically across the arc A by means of the functional relation $f(1/\bar{z}) = 1/\overline{f(z)}$. Using a concept originating with Gross [4], we associate with an arbitrary point P of $|z| = 1$ three sets of points: the *cluster set* $C(P)$ of $f(z)$ at P, defined as the set of all values α which $f(z)$ approaches on a sequence of points of K converging to P; the *range of values* $R(P)$ of $f(z)$ at P, which consists of all the values α which $f(z)$ assumes infinitely often in every neighborhood of P; the *asymptotic set* $\Gamma(P)$ of $f(z)$ at P, which consists of all values α which $f(z)$ approaches along a Jordan arc which lies, except for one endpoint, entirely in K, and which terminates at the point P. In this connection we shall say that a value α in $\Gamma(P)$ is an *asymptotic value* of $f(z)$ at P.

The most interesting problem is that of the behavior of $f(z)$ in the neighborhood of P whenever P is a singular point of A; that is, whenever $f(z)$ cannot be continued analytically across any arc of $|z| = 1$ containing P. Among the most significant results concerning the behavior of meromorphic functions in the neighborhood of a singular point on $|z| = 1$, we mention the recent theorem of Carathéodory [2; 266] which states that the cluster set $C(P)$ for each singular point P on A is either one of the two sets $|w| \leq 1$, $|w| \geq 1$, or else the extended plane. Nevanlinna [8; 28] had previously shown that if $f(z)$ is analytic and bounded, $|f(z)| < 1$, in $|z| < 1$, and if $\lim_{r\to 1} |f(re^{i\theta})| = 1$ almost everywhere on an arc A of $|z| = 1$, then the radial limit values $f^*(e^{i\theta})$ of $f(z)$ in any neighborhood of a singular point P on A comprise a set on $|w| = 1$ of measure 2π. This result was improved by Seidel [11; 208], who showed that every point of $|w| = 1$ is a radial limit value of $f(z)$ in any neighborhood of P.

Our principal result, previously announced in [5], is that a meromorphic function having at most a finite number of zeros and poles in the region $0 < 1 - \varepsilon < |z| < 1$, $\theta_1 < \theta < \theta_2$, can be continued analytically beyond the arc A: $\theta_1 < \theta < \theta_2$ by means of the Schwarz reflection principle if and only if $f(z)$ admits neither 0 nor ∞ as an asymptotic value at any point of A. From this result follows an extension of the theorems of Nevanlinna and Seidel to the case of meromorphic functions of bounded characteristic. We prove first a theorem for meromorphic functions of bounded characteristic which extends a result of Seidel [11; 207] for bounded functions, and which provides the motivation for Theorem 2.

THEOREM 1. *Let* $f(z)$ *be meromorphic with bounded characteristic in* $|z| < 1$, *and let* $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ *exist with modulus* 1 *for almost all* $e^{i\theta}$ *belonging to an arc* A: $0 \leq \theta_1 < \theta < \theta_2 \leq 2\pi$ *of* $|z| = 1$. *If* $f(z)$ *has no zeros or poles in the region* $0 < 1 - \varepsilon < |z| < 1$, $\theta_1 < \theta < \theta_2$, *then the set of singularities of* $f(z)$ *on* A *is the closure on* A *of the set of points* $e^{i\theta}$ *for which* $f^*(e^{i\theta}) = 0$ *or* $f^*(e^{i\theta}) = \infty$.

As a function of bounded characteristic, $f(z)$ has a representation (see, e.g., Nevanlinna [7; 190])

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$$(1)\qquad f(z) = z^m \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z} \prod_{k=1}^{\infty} \frac{|b_k|}{\bar{b}_k} \frac{1 - \bar{b}_k z}{b_k - z} \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\mu(\phi) + i\lambda\right],$$

where the a_j and b_k are the zeros and poles, respectively, of $f(z)$, $\mu(\phi)$ is a function of bounded variation in $0 \leq \phi \leq 2\pi$, λ is a real constant, and m is an integer. Since the products in (1) are analytic with modulus 1 on A, it suffices to prove the theorem for the exponential factor in (1). If $f(z)$ were analytic on A, then $|f^*(e^{i\theta})|$ would have the value 1 at all points $e^{i\theta}$ of A. In this case the equations $f^*(e^{i\theta}) = 0$ and $f^*(e^{i\theta}) = \infty$ would have no solutions in A. Since any point of A for which either $f^*(e^{i\theta}) = 0$ or $f^*(e^{i\theta}) = \infty$ is a singular point for $f(z)$, it is clear that the set of singularities of $f(z)$ on A contains the closure of the set of points for which $f^*(e^{i\theta}) = 0$ or $f^*(e^{i\theta}) = \infty$.

We shall show next that, if P is a singular point of $f(z)$ on A, then $f^*(P) = 0$ or ∞ or else the solutions of the equations $f^*(e^{i\theta}) = 0$ and $f^*(e^{i\theta}) = \infty$ have P as a limit point. Let E be an arbitrarily small subarc of A with P in its interior, and let $\bar{E}$ be the complementary arc on $|z| = 1$ defined by E. Now if $\mu(\phi)$ were constant on E, (1) would reduce to

$$(2)\qquad f(z) = e^{i\lambda} \exp\left[\frac{1}{2\pi} \int_{\bar{E}} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\mu(\phi)\right].$$

It would follow from (2) that $f(z)$ is analytic on E, and *a fortiori* at P; hence $\mu(\phi)$ cannot be constant on E. In the Lebesgue decomposition of $\mu(\phi)$ on E, we have

$$(3)\qquad \mu(\phi) = \nu(\phi) + g(\phi) + \psi(\phi),$$

where $\nu(\phi)$ is absolutely continuous with $\nu'(\phi) = \mu'(\phi)$ almost everywhere, $g(\phi)$ is continuous and of bounded variation with $g'(\phi) = 0$ almost everywhere, and $\psi(\phi)$ is a step-function. Since $|f^*(e^{i\theta})| = 1$ almost everywhere on E, it follows that $\nu(\phi)$ is identically constant on A. If $\psi(\phi)$ is not constant on E, it must have a jump in every neighborhood of P; otherwise, we could find a smaller arc E' containing P such that $\psi(\phi)$ is constant in E'. It is known [6; 244] that at each jump of $\psi(\phi)$, $\mu'(\phi) = \pm\infty$, depending on whether the saltus of $\psi(\phi)$ is positive or negative. At such jumps, either $f^*(e^{i\theta}) = 0$ or $f^*(e^{i\theta}) = \infty$, so that P is again a limit point of the points for which $f^*(e^{i\theta}) = 0$ or $f^*(e^{i\theta}) = \infty$. Finally, if $g(\phi)$ is not identically constant on E, then, by a well-known theorem (see, e.g., Saks [10; 128]), $g'(\phi) = \pm\infty$ on a non-denumerable set of points of E. Such a set contains at least one point ϕ_0 for which $\mu'(\phi_0) = \pm\infty$. Since E is arbitrarily small, it follows that either $\mu'(P) = \pm\infty$ or else the points for which $\mu'(\phi) = \pm\infty$ have P as a limit point, which is a contradiction, since, if ϕ_0 is such that $\mu'(\phi_0) = \pm\infty$, then $f^*(e^{i\phi_0}) = 0$ or ∞. Hence $g(\phi)$ must be constant on E. This proves Theorem 1.

Frostman [3: 2] showed that there exists a function $f(z)$, analytic and bounded, $|f(z)| < 1$, in $|z| < 1$, with $|f^*(e^{i\theta})| = 1$ almost everywhere on $|z| = 1$, with the property that $f(z)$ is singular in $P = 1$, and such that $f^*(e^{i\theta})$ vanishes nowhere. The function $f(z)$ has an infinite number of zeros in $|z| < 1$, however, and these zeros have the point 1 as a limit point.

To see that Theorem 1 fails to hold if we drop the condition that $f(z)$ be of bounded characteristic, we observe that the function

$$(4) \qquad u(r, \theta) = \frac{\partial}{\partial\theta}\left\{\frac{1 - r^2}{1 + r^2 - 2r\cos\theta}\right\} = \frac{-2r(1 - r^2)\sin\theta}{[1 + r^2 - 2r\cos\theta]^2}, \quad z = re^{i\theta},$$

is harmonic in $|z| < 1$, with $\lim_{r\to 1} u(r, \theta) = 0$ for all values of θ. If $v(r, \theta)$ is a conjugate harmonic function of $u(r, \theta)$, the function

$$f(z) = e^{u+iv}$$

is analytic in $|z| < 1$, and $|f(z)| = e^u \to 1$ on all radii of $|z| < 1$. From the fact that

$$\int_0^{2\pi} |u(r, \theta)| d\theta$$

is not bounded independently of r, it follows that $f(z)$ is not of bounded characteristic in $|z| < 1$. The point $z = 1$ is obviously a singularity for $f(z)$, and since $\lim_{r\to 1} f(re^{i\theta})$, whenever it exists, is of modulus 1, it is clear that Theorem 1 does not apply to $f(z)$. It will be noticed, however, that, as z approaches 1 along the oricycles $r = \cos\theta$ $(\theta > 0)$ and $r = \cos\theta$ $(\theta < 0)$, $u(r, \theta) \to \pm\infty$, so that, along these paths, $f(z) \to 0$ or $f(z) \to \infty$, according as $\theta > 0$ or $\theta < 0$.

THEOREM 2. *Let* $f(z)$ *be meromorphic in* $|z| < 1$, *and let* $A: 0 \le \theta_1 < \theta < \theta_2 < 2\pi$ *be an arc of* $|z| = 1$ *such that, for almost all* $e^{i\theta}$ *on* A, $\lim_{r\to 1} |f(re^{i\theta})| = 1$. *If* $f(z)$ *has no zeros or poles in the region* $0 \le 1 - \varepsilon < |z| < 1$, $\theta_1 < \theta < \theta_2$, *then a necessary and sufficient condition that* $f(z)$ *can be continued analytically beyond* A *is that* $f(z)$ *admits neither* 0 *nor* ∞ *as asymptotic values on* A.

Let M be the set of points ζ on A such that 0 or ∞ belongs to $\Gamma(\zeta)$. We shall show that the set of singularities of $f(z)$ on A is the closure of M on A. Now if $f(z)$ is analytic on A, then $|f(z)| = 1$ for all points of A. Hence it follows that $\bar{M}$ is contained in the set of singularities of $f(z)$ on A.

We must show, conversely, that if P is a singular point of $f(z)$ on A, then P belongs to $\bar{M}$. Denote by $V = V(P, \delta)$ the circle $|z - P| < \delta$, where $\delta > 0$ is chosen small enough so that $f(z)$ has no zeros or poles in $\bar{V} \cap K$, and small enough so that the circle $|z - P| = \delta$ intersects A in two points. Letting $\rho > 0$ be a fixed real number less than 1, we form the open sets

$$(5) \qquad \begin{aligned} O_\rho &= V(P, \delta) \cap \{z \mid |z| < 1;\ |f(z)| < \rho\}, \\ Q_\rho &= V(P, \delta) \cap \{z \mid |z| < 1;\ |f(z)| > 1/\rho\}. \end{aligned}$$

It cannot happen that there exist values of δ and ρ for which both O_ρ and Q_ρ are empty; otherwise, for these values of δ and ρ, we would have in $V(P, \delta)$

$$\rho \le |f(z)| \le 1/\rho.$$

This relation, however, is not compatible with the result of Carathéodory mentioned in the introduction, since the cluster set $C(P)$ must be either the extended plane or one of the sets $|w| \le 1$ and $|w| \ge 1$. Hence at least one of O_ρ and Q_ρ must be non-empty for all δ and all ρ; we shall assume it to be O_ρ.

It follows from the maximum modulus theorem that every component G_k of O_ρ is simply-connected. We assert next that there exists at least one component G_k of O_ρ such that $\bar{G}_k \cap A$ is not empty. Consider the two subsets $\{G'_n\}$, $\{G''_m\}$ of the components G_k, those in the first class having one or more arcs of $|z - P| = \delta$ as a part of their frontiers, and those in the second class having no such arcs. It is an easy consequence of the minimum modulus theorem that $\bar{G}''_m \cap A$ is not empty. We show then that if the class $\{G''_m\}$ is empty, at least one G'_n has the desired property. By virtue of Carathéodory's theorem there exists a sequence $\{z_k\}$, $|z_k| < 1$, $z_k \to P$, with the property that $f(z_k) \to 0$. It is clear that we may assume that there exists an infinite subset $\{G'_{n_j}\}$ containing the z_k, since the existence of a finite subset yields our assertion immediately. From this subset we can extract a subsequence (which we relabel $\{G'_n\}$) whose frontiers converge to a limiting continuum C which connects P with a point of $|z - P| = \delta$. The continuum C cannot contain as a subset any sub-arc of A, since $|f(z)| = \rho < 1$ at all points of C interior to $|z| < 1$ and $|f(z)| \to 1$ almost everywhere on A. Now it is not difficult to show that C cannot be a limiting continuum of an infinite number of frontiers of the $\{G'_n\}$ without violating at some point of $|z| < 1$ the principle of the preservation of neighborhoods. Hence we can assume that there is at least one G_k with the property that $E_k = \bar{G}_k \cap A$ is not empty. In the sequel we shall speak only of components whose frontiers have a non-empty intersection with A, and we shall assume that the components have been relabelled G_k. We remark that $P \in E_k$. We need a lemma:

LEMMA. *For each* ρ *and each* n, *the frontier* Fr G_n *of* G_n *is locally-connected, and the linear measure of* E_n = Fr $G_n \cap A$ *is zero.*

This lemma can be proved by modifying slightly an earlier method of the author [6; 249]; we omit the details.

To complete the proof of Theorem 2, we consider two different values of $\rho: 0 < \rho_1 < \rho_2 < 1$. Let $G(\rho_1)$ be a component of O_{ρ_1} such that $\overline{G(\rho_1)} \cap A$ is not empty, and let G be that component of O_{ρ_2} which contains $G(\rho_1)$. Let $z = \psi(w)$ be a univalent analytic function which maps $|w| < 1$ conformally onto G, and let E_w be the set of points on $|w| = 1$ whose image under $z = \psi(w)$ is $E_z = \bar{G} \cap A$. Since the linear measure of E_z is zero, an extension, due to Ostrowski [9; 422], of the Löwner-Montel lemma implies that the linear measure of E_w is zero. We consider the function $F(w) = f[\psi(w)]/\rho_2$, which is analytic and bounded, $|F(w)| < 1$, in $|w| < 1$, and which has the property that $|F^*(e^{is})| = 1$ almost everywhere on an arc A' of $|w| = 1$, where $w = |w|e^{is}$. Let P' denote the image on A' of P; we remark that A' contains P'. If F(w) were analytic at P', we could find a circular neighborhood of P' with the property that in this neighborhood $1 - |F(w)| < 1 - \rho_1/\rho_2$. This implies that $|F(w)| > \rho_1/\rho_2 > \rho_1$, in some neighborhood of P', and that, under the mapping $\psi(w)$, $|f(z)| > \rho_1$ in some neighborhood of P. This contradicts the structure of $G(\rho_1)$ and proves that P' is a singular point for F(w). Since $F(w) \neq 0$ in $|w| < 1$, it follows from a well-known theorem of Seidel [11; 207] that there exists at least one radius $s = s_0$ such that $F^*(e^{is_0}) = 0$. This radius is mapped by $z = \psi(w)$ onto an arc L of G terminating at a point η of Fr G such that $f(z) \to 0$ as $z \to \eta$ along L. Since η cannot be an interior point of $|z| < 1$, η must be a point of A. An identical argument can be used if Q_ρ is not empty to show that there exists a path L' and a point η' of A such that $f(z) \to \infty$ as $z \to \eta'$ along L'. Since δ can be chosen arbitrarily small, it follows that $P \in \bar{M}$, and Theorem 2 is proved.

As an application of Theorem 2, we extend the result of Nevanlinna and Seidel mentioned in the introduction.

THEOREM 3. *Let* f(z) *be meromorphic with bounded characteristic in* $|z| < 1$, *and let* $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ *have modulus* 1 *for almost all* $e^{i\theta}$ *belonging to some arc* $A: \theta_1 < \theta < \theta_2$, *of* $|z| = 1$. *If* P *is a singular point of* f(z) *on* A, *then every* *value of modulus* 1 *which is not in the range* R(P) *of* f(z) *at* P *is an asymptotic value of* f(z), *at some point of each subarc of* A *containing the point* P.

Let us assume that $\zeta = e^{i\lambda}$ is not in the range R(P) of f(z) at P; then the function

$$\Phi(z) = \exp\left[\frac{f(z) + \zeta}{f(z) - \zeta}\right] \tag{6}$$

is analytic without zeros in the region $V(P, \delta) \cap K$ for sufficiently small $\delta > 0$. Since f(z) has bounded characteristic in $|z| < 1$, it follows from a simple corollary of the Riesz-Nevanlinna theorem [7; 197] that $\lim \inf_{r \to 1} |f(re^{i\theta}) - \zeta| > 0$ for almost all $e^{i\theta}$ on $|z| = 1$. Hence the function $\Phi(z)$ of (6) has the property that $\lim_{r \to 1} |\Phi(re^{i\theta})| = 1$ for almost all $e^{i\theta}$ on that arc of $|z| = 1$ which bounds $V(P, \delta) \cap K$. It follows trivially from the theorem of Carathéodory mentioned above that P is a singular point for $\Phi(z)$. Indeed, since the cluster set C(P) consists of one or both of the sets $|w| \leq 1$ and $|w| \geq 1$, either 0 or ∞ (or both) must be a cluster value of f(z) at P; it is convenient to assume that $0 \in C(P)$. This means that there exists a sequence $\{z_k\}$, $|z_k| < 1$, $\lim_{k \to \infty} z_k = P$, such that $\lim_{k \to \infty} f(z_k) = 0$. For this sequence we must have $\lim_{k \to \infty} \Phi(z_k) = e^{-1}$, which shows that $\Phi(z)$ cannot be regular at $z = P$ since $\lim_{r \to 1} |\Phi(re^{i\theta})| = 1$ almost everywhere. We may now apply Theorem 2 to the function $\Phi(z)$; it follows that there exists an arc L of $|z| < 1$, terminating either at P or at a point P' of $|z| = 1$ as close to P as we please, such that $\Phi(z)$ tends to 0 or ∞ as $z \to P'$ (or P) along L. This shows that f(z) admits $\zeta = e^{i\lambda}$ as an asymptotic value, either at P or at a point P' of $|z| = 1$ arbitrarily close to P.

We remark that in such applications of Theorem 2 it is not necessary to assume —as we did in Theorem 3—that f(z) is of bounded characteristic. The essence of the proof of Theorem 3 is in the statement that the function $\Phi(z)$ in (6) has the property that $\lim_{r \to 1} |\Phi(re^{i\theta})| = 1$ for almost all $e^{i\theta}$ belonging to some arc of $|z| = 1$ containing the singular point P. In Theorem 3, we were able to use the Riesz-Nevanlinna theorem for functions of bounded characteristic to assert that

$$\lim_{r \to 1} \inf |f(re^{i\theta}) - \zeta| = 0$$

on at most a set of measure zero on $|z| = 1$. It appears from a recent extension by E. F. Collingwood[1] of a uniqueness theorem of Wolf [12; 383] that for meromorphic functions with unbounded characteristic the notion of category, rather than measure, becomes significant. In fact, unless additional assumptions are made about f(z), the function $\Phi(z)$ of (6) may not have the desired property; for in a recent letter to the author, W. Rudin has indicated that he has proved the following result.

THEOREM (Rudin). *Let* $\phi(z)$ *be continuous in* $|z| < 1$, *and let* E *be a set on* $|z| = 1$ *of first category. Then there exists a function* f(z), *analytic in* $|z| < 1$, *such that for every* $e^{i\theta} \in E$, $\lim_{r \to 1}[f(re^{i\theta}) - \phi(re^{i\theta})] = 0$.

This theorem implies the existence of a function g(z), analytic in $|z| < 1$, with the property that $\lim_{r \to 1} \Re g(re^{i\theta}) = \infty$ and $\lim_{r \to 1} \Im g(re^{i\theta}) = 0$ for almost all $e^{i\theta}$ on $|z| = 1$. If we set $f(z) = [g(z) + 1]/[g(z) - 1]$, we have that $\lim_{r \to 1} f(re^{i\theta}) = 1$ almost everywhere on $|z| = 1$. The function $\Phi(z)$ of (6) then has the property that $\lim_{r \to 1} |\Phi(re^{i\theta})| = \lim_{r \to 1} \exp \Re g(re^{i\theta}) = \infty$ for almost all $e^{i\theta}$, so that the method

1. To appear in Acta Math. For a uniqueness theorem of a related character, see also [1].

of proof of Theorem 3 is not, in general, applicable to functions of unbounded characteristic. Whether Theorem 3 can be carried over to the case where f(z) is not of bounded characteristic is still an open question.

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