THE SPECIAL HOMOTOPY ADDITION THEOREM

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The homotopy addition theorem is an elementary result concerning homotopy groups which is used in proving the Hurewicz isomorphism theorem. It was long considered obvious, and only recently has a proof, given by Sze-tsen Hu, found its way into the literature [3]. This proof of the general theorem entails considerable complications of a technical nature, and it would be even more complicated if Hu did not define homotopy groups in an unusual way, using simplices instead of cubes as basic anti-images. The present paper attempts to avoid some of these complications by considering only that special case of the theorem which is actually needed in the proof of Hurewicz's theorem. Further, it follows the present-day trend by retaining the usual definition of homotopy groups but using the cubic rather than the simplicial singular homology.

Eilenberg's proof of the Hurewicz theorem ([1], pp. 439-444) is readily adapted to the cubic theory, but the (implicit) appeals to the homotopy addition theorem remain. The two final corollaries of the present paper apply the special homotopy addition theorem to this situation.

LEMMA 1. *Let* $u \in F^n(X, x)$ *and*

$$v(x_1, \cdots, x_n) = u(x_1, \cdots, x_{i-1}, 1 - x_{i+1}, x_i, x_{i+2}, \cdots, x_n).$$

Then $v \in F^n(X, x)$ *and* $u \simeq v$. (The notation is defined in [5], pp. 72, 73.)

It is obvious that $v \in F^n$. The homotopy is constructed by twisting the face of I_n lying in the x_i, x_{i+1} -plane through an angle of $\pi/2$ as t goes from 0 to 1. Specifically, the homotopy is defined as follows:

$$H(x_1, \cdots, x_n, t) = u(x_1, \cdots, x_{i-1}, f(x_i, x_{i+1}, t), g(x_i, x_{i+1}, t), x_{i+2}, \cdots, x_n),$$

where

$$f(x, y, t) = (Aa + 1)/2,$$
$$g(x, y, t) = (Ab + 1)/2,$$
$$a(x, y, t) = (2x - 1)\cos \pi t/2 - (2y - 1)\sin \pi t/2,$$
$$b(x, y, t) = (2x - 1)\sin \pi t/2 + (2y - 1)\cos \pi t/2,$$
$$A(x, y, t) = \frac{\max(|2x - 1|, |2y - 1|)}{\max(|a|, |b|)}, \text{ unless } x = y = 1/2, \text{ in which case } A = 0.$$

It is a matter of computation to show that H is well-defined and continuous and provides an admissible homotopy between u and v. (The word 'admissible' is used to emphasize that $\dot{I}_n$ must remain at x during the homotopy.) A is discontinuous at $x = y = 1/2$ (where $|a| = |b| = 0$); but since A is bounded (indeed, $A \leq \sqrt{2}$), f and g are continuous there.

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Remark. This homotopy may be used to demonstrate that π_n is abelian for $n \geq 2$ (compare [5], p. 74). For if $f \in F^n$ $(n \geq 2)$, let Tf be the cube defined by

$$Tf(x_1, \cdots, x_n) = f(1 - x_1, 1 - x_2, x_3, \cdots, x_n).$$

A double application of Lemma 1 shows that $f \simeq Tf$. Hence if $u, v \in F^n(X, x)$, then $u \simeq Tu$ and $v \simeq Tv$, whence

$$u + v \simeq Tu + Tv \simeq T(Tu + Tv).$$

It is readily verified that $T(Tu + Tv) = v + u$.

COROLLARY. *If* $u, v \in F^n$, *let*

$$w(x_1, \cdots, x_n) = u(x_1, \cdots, 2x_i, \cdots, x_n) \text{ for } 0 \leq x_i \leq 1/2$$
$$w(x_1, \cdots, x_n) = v(x_1, \cdots, 2x_i - 1, \cdots, x_n) \text{ for } 1/2 \leq x_i \leq 1,$$

and denote w *by* $u +_i v$. *Then* $u +_i v \simeq u + v$.

This corollary merely states that the group operation in π_n is independent of which coordinate we choose to "add along." Assume $n > 1$, since otherwise the theorem is trivial. If $f \in F^n$, let

$$Tf(x_1, \cdots, x_n) = f(x_1, \cdots, x_{j+1}, 1 - x_j, \cdots, x_n)$$

and

$$Sf(x_1, \cdots, x_n) = f(x_1, \cdots, 1 - x_{j+1}, x_j, \cdots, x_n).$$

Then $f \simeq Tf$ and $f \simeq Sf$. If $u, v \in F^n$, then $v \simeq Tv$ and $u \simeq Tu$, so that $v +_j u \simeq Tv +_j Tu$ (this is proved just as for addition along the first coordinate); therefore $v +_j u \simeq S(Tv +_j Tu)$. But $S(Tv +_j Tu) = u +_{j+1} v$. Since this holds for $1 \leq j < n$, $u +_i v$ is homotopic either to $u + v$ or to $v + u$. Since π_n is abelian, this means that $u +_i v \simeq u + v$.

In considering cubic singular homology, we use the notation of [4], pp. 439, 440. We shall let $Q_{n,m}(X, x)$ denote the subgroup of $Q_n(X)$ generated by those singular cubes under which all faces of I_n of dimension less than m are mapped into x. Then $\Sigma_n Q_{n,m}$ corresponds to the group $S_m(X)$ used by Eilenberg ([1], p. 439).

LEMMA 2. *There exists a map* $f_i : I_n \to I_n$ *such that if* u *is a cube of* $Q_{n,n-1}(X, x)$ *and* $v = uf_i$, *then*

(1) v *is a cube of* $Q_{n,n-1}$,

(2) $\lambda_i^\varepsilon v \equiv x$ $(\varepsilon = 0, 1)$,

(3) $\lambda_{i+1}^0 v \simeq (\lambda_{i+1}^0 u) +_i (\lambda_i^1 u)$,

(4) $\lambda_{i+1}^1 v \simeq (\lambda_i^0 u) +_i (\lambda_{i+1}^1 u)$,

(5) $\lambda_j^\varepsilon v \simeq \lambda_j^\varepsilon u$ *(as elements of* $F^{n-1}(X, x)$*), if* $j \neq i, i + 1$; *and if* $\lambda_j^\varepsilon u \equiv x$, *then* $\lambda_j^\varepsilon v \equiv x$.

Proof. We define a map $F : I_3 \to I_2$ by defining it first on $I_2 \times 0$ and $\dot{I}_2 \times I$ and then using the homotopy extension theorem ([2], p. 20) to extend it to all of I_3. Let

(a) $F(x, y, 0) = (x, y)$,

(b) $F(x, 0, t) = \left(\frac{2x}{2 - t}, 0\right)$ for $0 \leq x \leq 1 - t/2$,

$F(x, 0, t) = \left(1, \frac{2x}{2 - t} - 1\right)$ for $1 - t/2 \leq x \leq 1$,

(c) $F(1, y, t) = \left(1, \frac{2y + 2}{2 - t} - 1\right)$ for $0 \leq y \leq 1 - t$,

$F(1, y, t) = (1, 1)$ for $1 - t \leq y \leq 1$,

(d) $F(x, 1, t) = (1, 1) - F(1 - x, 0, t)$,

(e) $F(0, y, t) = (1, 1) - F(1, 1 - y, t)$.

It is readily verified that definitions (b) and (c) are unambiguous on their respective domains. Then each of (a) to (e) defines F on a face of I_3; it remains only to check that the definitions agree on the intersection of these faces. Note also that F maps $\dot{I}_2 \times I$ into $\dot{I}_2$ and $I_2 \times 0$ into I_2, so that it may be extended to map I_3 into I_2.

Now let a and b be functions such that $F(x, y, t) = (a(x, y, t), b(x, y, t))$. Then a and b have the following properties:

(A) $a(x, y, 0) = x$ and $b(x, y, 0) = y$.

(B) $a(\varepsilon, y, 1) = b(\varepsilon, y, 1) = \varepsilon$ for $\varepsilon = 0$ and $\varepsilon = 1$.

(C) If either x or y is 0 or 1, either a or b must be 0 or 1 (since $F : \dot{I}_2 \times I \to \dot{I}_2$).

(D) $a(x, 0, 1) = b(x, 1, 1) = 2x$ for $0 \leq x \leq 1/2$,

$a(x, 0, 1) = b(x, 1, 1) = 1$ for $1/2 \leq x \leq 1$,

$a(x, 1, 1) = b(x, 0, 1) = 0$ for $0 \leq x \leq 1/2$,

$a(x, 1, 1) = b(x, 0, 1) = 2x - 1$ for $1/2 \leq x \leq 1$.

Now we define $f_i : I_n \to I_n$ as follows:

(*) $$f_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, a(x_i, x_{i+1}, 1), b(x_i, x_{i+1}, 1), \cdots, x_n),$$

and verify that f_i has the desired properties. Let u be a cube of $Q_{n,n-1}$, and let $v = uf_i$.

Property (1). v is a cube of $Q_{n,n-1}$. Suppose that two of the variables $x_1, \cdots, x_n$ of v are 0 or 1. Then one may use (B) and (C) above to check that in every possible case at least two of the n quantities on the right side of (*) are 0 or 1. Since u is a cube of $Q_{n,n-1}$, it follows that $v \equiv x$, as desired.

Properties (2) to (4) are easy consequences of (B) and (D). To prove (5), suppose that $j \neq i, i + 1$ $(1 \leq j \leq n)$. Let

(**) $$w(x_1, \cdots, x_{n+1}) = u(x_1, \cdots, a(x_i, x_{i+1}, x_{n+1}), b(x_i, x_{i+1}, x_{n+1}), \cdots, x_n),$$

and define

$$H(x_1, \cdots, x_n) = \lambda_j^{\varepsilon} w.$$

Then H maps $I_{n-1} \times I$ continuously into X; and $H(x_1, \cdots, 0) = \lambda_j^{\varepsilon} u$ by (A) (since $j \leq n$), while $H(x_1, \cdots, 1) = \lambda_j^{\varepsilon} v$ by definition of v. Moreover, H maps $\dot{I}_{n-1} \times I$ into x, for H is obtained from (**) by setting $x_j = \varepsilon$ and shifting indices, and if one of the variables $x_1, \cdots, x_{n-1}$ of H is 0 or 1, it follows from (C) that another one, besides x_j, of the quantities in the right side of (**) is 0 or 1. Then since u is a cube of $Q_{n,n-1}$, $H \equiv x$. Hence H is an admissible homotopy between $\lambda_j^{\varepsilon} u$ and $\lambda_j^{\varepsilon} v$. Moreover, if $\lambda_j^{\varepsilon} u \equiv x$, $H \equiv x$ from the definition, so that $\lambda_j^{\varepsilon} v \equiv x$.

THEOREM (The special homotopy addition theorem; compare [3], p. 118). *Let* u *map* I_{n+1} *into* X *and all faces of* I_{n+1} *of dimension less than* n *into* x. *Then* $\lambda_j^{\varepsilon} u \in F^n(X, x)$ *and*

$$\Sigma_1^{n+1} (-1)^i \lambda_i^0 u - \Sigma_1^{n+1} (-1)^i \lambda_i^1 u = 0.$$

Proof. By hypothesis, u is a cube of $Q_{n+1,n}$. Let

$$v_i = u f_1 f_2 \cdots f_i \quad (i \leq n),$$

where the f_j are the functions of Lemma 2. If $n = 1$, then $v_1 : I_2 \to X$ and by (2) of Lemma 2, $\lambda_1^{\varepsilon} v_1 \equiv x$, so that v_1 is an admissible homotopy between

$$\lambda_2^0 v_1 = \lambda_2^0 u + \lambda_1^1 u \text{ and } \lambda_2^1 v_1 = \lambda_1^0 u + \lambda_2^1 u,$$

by virtue of (3) and (4). Hence $-\lambda_1^0 u + \lambda_2^0 u + \lambda_1^1 u - \lambda_2^1 u = 0$, as desired.

Suppose $n > 1$. Then $v_i : I_{n+1} \to X$, and by induction we may show that

$$\text{(I)} \qquad \lambda_j^{\varepsilon} v_i \equiv x \text{ for } j \leq i$$

and

$$\text{(II)} \qquad \lambda_{i+1}^{\varepsilon} v_i = \Sigma_{i+1}^1 \lambda_j^{\varepsilon_j} u,$$

where $\varepsilon_{i+1} = \varepsilon$ and the other superscripts alternate between 0 and 1. These properties hold trivially for $i = 0$; suppose they hold for $i - 1$. By the induction hypothesis, $\lambda_j^{\varepsilon} v_{i-1} \equiv x$ for $j \leq i - 1$, whence by property (5) of Lemma 2, $\lambda_j^{\varepsilon} v_i \equiv x$ for $j \leq i - 1$. On the other hand, (1) states that $\lambda_i^{\varepsilon} v_i$ is also identically equal to x. Hence (I) holds.

To verify (II), we use (3) and (4), apply the corollary of Lemma 1 to change the symbol $+_i$ to $+$, and recall that π_n is abelian. We get the relation

$$\lambda_{i+1}^{\varepsilon} v_i = \lambda_{i+1}^{\varepsilon} v_{i-1} + \lambda_i^{\delta} v_{i-1},$$

where $\delta = 1 - \varepsilon$. Now by (5), $\lambda_{i+1}^{\varepsilon} v_{i-1} = \lambda_{i+1}^{\varepsilon} u$, whence

$$\lambda_{i+1}^{\varepsilon} v_i = \lambda_{i+1}^{\varepsilon} u + (\lambda_i^{\varepsilon_i} u + \cdots + \lambda_1^{\varepsilon_1} u),$$

where $\delta = \varepsilon_i$, by the induction hypothesis. Then (I) and (II) hold for all $i \leq n$.

By (I), v_n maps I_{n+1} into X and $\lambda_j^{\varepsilon} v_n = x$ for all $j \leq n$. Hence v_n is an admissible homotopy between $\lambda_{n+1}^0 v_n$ and $\lambda_{n+1}^1 v_n$, so that $\lambda_{n+1}^0 v_n - \lambda_{n+1}^1 v_n = 0$. By virtue of property (II), this is merely a restatement of the conclusion of the theorem.

The theorem above enters into the proof of Hurewicz's theorem by way of two corollaries, which pertain to pages 442 and 443 of [1], respectively.

We have the natural map of $F^n(X, x) \to Q_n(X)$ which assigns to $f \in F^n$ the same map, considered as a singular cube and denoted by u_f. It is shown that the homology class of u_f depends only on the homotopy class of f, so that a map of π_n into H_n is induced.

COROLLARY 1. *The natural map* $\pi_n \to H_n$ *is a homomorphism.*

Let $f, g \in F^n$ and let h denote $f + g \in F^n$. We wish to show that u_h is homologous to $u_f + u_g \in Q_n$. We define $G : I_{n+1} \to X$ as follows:

$$G(0, x_2, x_3, \cdots, x_{n+1}) = f(x_2, \cdots, x_{n+1}),$$
$$G(x_1, 1, x_3, \cdots, x_{n+1}) = g(x_1, x_3, \cdots, x_{n+1}),$$
$$G \equiv x \text{ elsewhere on } I \times \dot{I}_n .$$

Then G is defined on $(0 \times I_n) \cup (I \times \dot{I}_n)$ and may thus be extended to $I \times I_n$. Let $G(1, x_1, \cdots, x_n)$ be denoted by k or by u_k, depending on whether we consider it in F^n or Q_n. Now $f = \lambda_1^0 G$, $g = \lambda_2^1 G$, $k = \lambda_1^1 G$ and $G \equiv x$ on all the other faces of I_{n+1}. This gives us two results: first, since $dG = -u_f - u_g + u_k$ (the other cubes of the sum are degenerate), u_k is homologous to $u_f + u_g$. On the other hand, the previous theorem states that $-f + k - g = 0$, so that $k = f + g = h$. Then, by a previous remark, u_k and u_h determine the same homology class. Hence u_h is homologous to $u_f + u_g$, and the proof is complete.

We may define a map π of $Q_{n,n}(X, x)$ into $\pi_n(X, x)$ merely by assigning to every singular cube in $Q_{n,n}$ its homotopy class, and extending linearly. If $n = 1$, we take π_1/C instead of π_1, where C is the commutator subgroup of π_1. If T is a singular cube in $Q_{n+1,n}$, then $dT \in Q_{n,n}$ and

COROLLARY 2. $\pi(dT) = 0$.

$dT = \Sigma_1^{n+1}(-1)^i(\lambda_i^0 T - \lambda_i^1 T)$. This expression is merely a formal addition of singular cubes, but $\pi(dT)$ is the homotopy class of the sum of these cubes, where the sum is taken in F^n. This sum differs from the expression of the previous theorem only in the arrangement of the terms, so that it is either homotopic to zero (if $n > 1$) or to an element of C (if $n = 1$).

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