ON THE STRUCTURE OF RECURRENCE RELATIONS II

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A previous note [1] was concerned with necessary conditions for the existence of recurrence relations between contiguous solutions of an equation

$$(1) \qquad d/dZ\,[P(Z)dY/dZ] + Q(Z, t)Y = 0,$$

for the special case $Q(z, t) = R(Z) + tS(Z)$. For each m, let $Y_m^{(1)}$ and $Y_m^{(2)}$ constitute a fundamental system of solutions of (1) corresponding to $t = t_m$. It was proved in [1] that a pair of first-order differential recurrence relations, valid both for

$$Y_n = Y_n^{(1)},\ Y_{n-1} = Y_{n-1}^{(1)} \text{ and for } Y_n = Y_n^{(2)},\ Y_{n-1} = Y_{n-1}^{(2)},$$

can always be put in the form

$$(2) \qquad P(Z)\,A_n(Z)\,dY_n/dZ + B_n(Z)\,Y_n = Y_{n-1},$$

$$(3) \qquad P(Z)\,A_n(Z)\,dY_{n-1}/dZ + C_n(Z)\,Y_{n-1} = d_n\,Y_n.$$

(As in [1], capital letters denote functions of Z, and lower-case letters denote quantities independent of Z.) Relations of the form (2) and (3) which are valid for two fundamental systems will be called *general recurrence relations*, and relations similar to (2) and (3) which are valid for a single pair of solutions Y_n, Y_{n-1}—but for no other pair linearly independent of these solutions—will be called *particular recurrence relations*.

In this note the problem of sufficiency will be solved, for quite general $Q(Z,t)$. It will be shown that the solutions of (1) corresponding to $t = t_n$ and to $t = t_{n-1}$ always satisfy a pair of general recurrence relations (2), (3), with coefficients that are unique for a fixed normalization of the solutions; that in addition, corresponding to each pair of particular solutions Y_n, Y_{n-1}, there exists a family of particular recurrence relations depending on two arbitrary functions of Z; and that these exhaust all possibilities.

Here—as also in [1]—the emphasis has been placed on a pair of contiguous solutions Y_n, Y_{n-1} only because relations among these are in general of greatest interest. It will be easy to see that all results apply equally well to the solutions corresponding to an arbitrary pair t', t'' of parameter values in (1), or indeed (as was suggested by G. Y. Rainich) to the solutions of any pair of second-order linear differential equations after a suitable transformation (e.g., if $p \equiv 1$ in both equations).

For the present, two fixed sequences $\{Y_m^{(1)}\}$, $\{Y_m^{(2)}\}$ of fundamental systems of solutions are assumed given, with each solution normalized in some appropriate manner. As is well known, the Wronskian of the fundamental system has for each m the value

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(4) $$W_m = Y_m^{(1)}\, dY_m^{(2)}/dZ - dY_m^{(1)}/dZ\, Y_m^{(2)} = 1/[h_m P(Z)].$$

with h_m depending only on m. If the solutions for $t = t_n$ and $t = t_{n-1}$ possess a pair (2) and (3) of general recurrence relations, an identity relating A_n and d_n to the Wronskians (4) for $m = n$ and $m = n - 1$ is easily calculated:

(5) $$P\, A_n \left[Y_n^{(1)}\, dY_n^{(2)}/dZ - dY_n^{(1)}/dZ\, Y_n^{(2)}\right] = Y_n^{(1)}\, Y_{n-1}^{(2)} - Y_{n-1}^{(1)}\, Y_n^{(2)},$$

(6) $$P\, A_n \left[Y_{n-1}^{(1)}\, dY_{n-1}^{(2)}/dZ - dY_{n-1}^{(1)}/dZ\, Y_{n-1}^{(2)}\right] = d_n \left[Y_{n-1}^{(1)}\, Y_n^{(2)} - Y_n^{(1)}\, Y_{n-1}^{(2)}\right],$$

so that

(7) $$d_n = -\, W_{n-1}/W_n = -\, h_n/h_{n-1}.$$

Write equation (2), once with $Y_n = Y_n^{(1)}$, $Y_{n-1} = Y_{n-1}^{(1)}$, and a second time with $Y_n = Y_n^{(2)}$, $Y_{n-1} = Y_{n-1}^{(2)}$, and eliminate the right-hand sides from these two equations to obtain an identity relating B_n to A_n. From equation (3), obtain similarly an identity relating C_n to A_n:

(8) $$P\, A_n \left[Y_{n-1}^{(1)}\, dY_n^{(2)}/dZ - dY_n^{(1)}/dZ\, Y_{n-1}^{(2)}\right] + B_n \left[Y_{n-1}^{(1)}\, Y_n^{(2)} - Y_n^{(1)}\, Y_{n-1}^{(2)}\right] = 0,$$

(9) $$P\, A_n \left[Y_n^{(1)}\, dY_{n-1}^{(2)}/dZ - dY_{n-1}^{(1)}/dZ\, Y_n^{(2)}\right] + C_n \left[Y_n^{(1)}\, Y_{n-1}^{(2)} - Y_{n-1}^{(1)}\, Y_n^{(2)}\right] = 0.$$

It is now quite elementary to describe the structure of general recurrence relations:

THEOREM 1. *For every fixed pair of sequences* $Y_m^{(1)}$, $Y_m^{(2)}$ *of fundamental systems of solutions of the second-order linear differential equation* (1), *there exists a unique pair of general recurrence relations* (2) *and* (3).

Proof: First it is necessary to show that the right-hand members of (5) and (6) cannot be identically zero. Suppose $Y_n^{(1)}\, Y_{n-1}^{(2)} - Y_{n-1}^{(1)}\, Y_n^{(2)} = 0$ for all Z. Rearrangement and differentiation give the identity

$$\frac{Y_{n-1}^{(1)}\, dY_{n-1}^{(2)}/dZ - dY_{n-1}^{(1)}/dZ\, Y_{n-1}^{(2)}}{\left[Y_{n-1}^{(2)}\right]^2} = \frac{Y_n^{(1)}\, dY_n^{(2)}/dZ - dY_n^{(1)}/dZ\, Y_n^{(2)}}{\left[Y_n^{(2)}\right]^2}$$

or, by virtue of (4), $\left[Y_{n-1}^{(2)}\right]^2 / \left[Y_n^{(2)}\right]^2 = h_n/h_{n-1}$, so that $Y_{n-1}^{(2)}$ is a constant multiple of $Y_n^{(2)}$. Together with the original identity in the guise $Y_n^{(2)}/Y_{n-1}^{(2)} = Y_n^{(1)}/Y_{n-1}^{(1)}$, this result is a contradiction of the hypothesis that the functions of index n and those of index n - 1 are solutions of distinct equations of the type (1).

The existence of recurrence relations now follows from the fact that the formulas (4), (7), and

(10) $$A_n = h_n\left[Y_n^{(1)}\, Y_{n-1}^{(2)} - Y_{n-1}^{(1)}\, Y_n^{(2)}\right],$$

(11) $$B_n = h_n\, P\left[Y_{n-1}^{(1)}\, dY_n^{(2)}/dZ - dY_n^{(1)}/dZ\, Y_{n-1}^{(2)}\right],$$

(12) $$C_n = -h_n\, P\left[Y_n^{(1)} dY_{n-1}^{(2)}/dZ - dY_{n-1}^{(1)}/dZ\, Y_n^{(2)}\right],$$

reduce (2) and (3) to identities. Formula (10) is obtained from (4) and (5), formula (11) from (4), (5), and (8), and formula (12) from (4), (5), and (9). If the coefficients were not unique, the difference of two distinct relations (2) would furnish a first-order differential equation having two linearly independent solutions, and similarly for (3).

More generally, the recurrence relations between $Y_m^{(1)}$ and $Y_n^{(1)}$ or $Y_m^{(2)}$ and $Y_n^{(2)}$, for arbitrary m and n, may now be written in a more symmetrical form. With the notation

$$U_{mn} = Y_m^{(1)}\, Y_n^{(2)} - Y_n^{(1)}\, Y_m^{(2)},\quad V_{mn} = Y_n^{(1)}\, dY_m^{(2)}/dZ - dY_m^{(1)}/dZ\, Y_n^{(2)},$$

and with W_m and W_n as in (4) and (7), one may rewrite relations analogous to (2) and (3) in the form

$$U_{mn}\, dY_m/dZ + V_{mn}\, Y_m = W_m\, Y_n,$$

$$U_{nm}\, dY_n/dZ + V_{nm}\, Y_n = W_n\, Y_m.$$

Consider next a pair of particular solutions of equation (1), of the form

$$U_n = aY_n^{(1)} + bY_n^{(2)},\quad U_{n-1} = aY_{n-1}^{(1)} + bY_{n-1}^{(2)}.$$

Any pair of particular solutions corresponding to $t = t_n$, $t = t_{n-1}$, respectively, can be put in this form by a suitable choice of the fundamental systems of solutions. Let

(13) $$P(Z)\, E_n(Z)\, dU_n/dZ + F_n(Z)\, U_n = U_{n-1},$$

(14) $$P(Z)\, G_n(Z)\, dU_{n-1}/dZ + H_n(Z)\, U_{n-1} = d_n\, U_n$$

be a pair of particular recurrence relations for U_n, U_{n-1}. The structure of the coefficients is easily deduced:

THEOREM 2. *If* (13), (14) *is a pair of particular recurrence relations for* U_n, U_{n-1}, *then there exist functions* $K(Z)$ *and* $L(Z)$, *not both identically zero, such that*

(15) $$E_n = A_n + K\, U_n,\quad F_n = B_n - K\, P\, dU_n/dZ,$$

(16) $$G_n = A_n + L\, U_{n-1},\quad H_n = C_n - L\, P\, dU_{n-1}/dZ.$$

Conversely, for every arbitrary pair of functions $K(Z)$ *and* $L(Z)$, *not both* *identically zero, the equations* (13) *and* (14), *with coefficients as in* (15) *and* (16), *constitute a pair of particular recurrence relations for* U_n *and* U_{n-1}.

Proof: Suppose that (13) and (14) hold. The recurrence relations (2) and (3) are clearly valid for $Y_n = U_n$, $Y_{n-1} = U_{n-1}$. Subtract (2) with this choice of functions from (13): the resulting equation

$$P(E_n - A_n)dU_n/dZ + (F_n - B_n)\, U_n = 0$$

implies the existence of a function $K(Z)$ such that

$$E_n - A_n = K\, U_n,\quad F_n - B_n = -K\, PdU_n/dZ,$$

and (15) is proved. Similarly (16) is proved from equations (3) and (14), and K and L cannot both be zero since (13) and (14) are to hold for one pair of functions only. To prove the converse, note that if the coefficients (15) and (16) are substituted into the relations (13) and (14), these relations take the forms (2) and (3), respectively, with $Y_n = U_n$, $Y_{n-1} = U_{n-1}$ —independently of K and L.

In the special case where $E_n = G_n$, it is easy to see that

$$E_n = M(Z)\, U_n\, U_{n-1},\quad F_n = -M\, PdU_n/dZ\, U_{n-1},\quad G_n = -M\, P\, U_n\, dU_{n-1}/dZ,$$

so that only one arbitrary function appears. If M is chosen to be constant, one has essentially the case treated in [1], where the coefficient A_n of a general recurrence relation was proved to be of the form

$$A_n = a_n^{11} Y_n^{(1)} Y_{n-1}^{(1)} + a_n^{12} Y_n^{(1)} Y_{n-1}^{(2)} + a_n^{21} Y_n^{(2)} Y_{n-1}^{(1)} + a_n^{22} Y_n^{(2)} Y_{n-1}^{(2)}, \tag{17}$$

while B_n and C_n were derived from A_n by quadratures. As follows from Theorem 1, for general recurrence relations the choice of constants is

$$a_n^{11} = a_n^{22} = 0,\quad a_n^{12} = -a_n^{21} = h_n.$$

With a_n^{11} an arbitrary constant, $a_n^{22} = 0$, and $a_n^{12} = -a_n^{21} = h_n$, the function A_n in (17) would lead to a pair of particular recurrence relations for $Y_n^{(1)}$, and similarly for $Y_n^{(2)}$. If all four constants in (17) differ from zero, it is impossible to construct recurrence relations with A_n as coefficient.

An illustration is furnished by the recurrence relations for the modified Mathieu functions of the first kind, obtained by E. T. Whittaker and given in [2] along with two different representations of the coefficients. If Whittaker's formulas are rewritten in the standard form of this note, the coefficient corresponding to A_n is of the form (17), with $a_n^{11} \neq 0$ and with $a_n^{21} \neq -a_n^{12}$. It is clear from Theorem 1 that these are particular recurrence relations for the functions of the first kind alone. Comparison with equations (10) and (15) shows that

$$h_n = -a_n^{21},\quad K(Z) = a_n^{11}\, Y_{n-1}^{(1)} + \left[a_n^{12} + a_n^{21}\right] Y_{n-1}^{(2)}$$

(in the notations of equation (4) and Theorem 2). Removal of the term $K(Z)Y_n^{(1)}$ in Whittaker's coefficient corresponding to A_n, and appropriate changes in the coefficients corresponding to B_n and to C_n, would both simplify these coefficients and extend the validity of the recurrence relations to the Mathieu functions of the second kind.

For differential equations (1) for which the coefficients of a pair of general recurrence relations (2) and (3) are known in a representation different from the formulas (10), (11) and (12), the comparison of the two representations yields identities among the solutions. Examples of such identities among the prolate spheroidal functions will appear in [3].

REFERENCES

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2. E. T. Whittaker, *On the recurrence-formulae of the Mathieu functions*, J. London Math. Soc. 4 (1929), 88-96.

3. UMM-126, a forthcoming publication of the Willow Run Research Center of the University of Michigan.

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