LOCALLY PROJECTIVE SPACES OF DIMENSION ONE

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Let P be the real projective line, and let $\tilde{P}$ be the universal covering space of P with the lifted projective structure. Topologically, $\tilde{P}$ is an interval. Every projective mapping of a neighborhood in $\tilde{P}$ onto another neighborhood is the restriction of a unique projective homeomorphism of $\tilde{P}$ onto $\tilde{P}$. A projective transformation of P can be expressed with respect to homogeneous or nonhomogeneous preferred coordinates as follows:

homogeneous coordinates: $(X, Y) \to (X^*, Y^*) = (aX+bY, cX+dY)$, $ad-bc \neq 0$;

nonhomogeneous coordinates: $(X = xY)$ $x \to x^* = \dfrac{ax+b}{cx+d}$.

A *locally projective space* Z of dimension 1 is a manifold and a complete (that is, not properly contained in a larger) *atlas* of mutually compatible homeomorphisms, called *maps*, of neighborhoods in Z onto neighborhoods in $\tilde{P}$. Two maps $f: U \to U'$ and $g: V \to V'$ are *compatible* if, for each connected component W of the intersection $U \cap V$, the mapping gf^{-1} restricted to $f(W)$ is a projective transformation in $\tilde{P}$.

Two maps in the atlas, both covering the point z in Z, are said to be equivalent at z if their restrictions to some neighborhood of z coincide. Following Ehresmann [2], we call an equivalence class of maps in the atlas, all covering z, *a local jet*, and we denote it by j_z. A topology in the set of jets can be introduced by giving a base for the open sets as follows: the set of jets (that is, of equivalence classes of maps) which contain a map is an open set in the space of jets.

A connected component of the space of jets is a covering space of Z, with the projection $j: j_z \to z$. The mapping which sends a jet j_z of this covering space into the image in $\tilde{P}$ of the point z under each of the maps of the equivalence class j_z is a homeomorphism onto an interval in $\tilde{P}$. This interval can therefore be considered as the universal covering space $\tilde{Z}$ with lifted locally projective structure of Z.

An analogous theorem and proof, given in [2], exist in the case of the locally homogeneous spaces, where the homogeneous space with a transitive Lie group of transformations takes the place of the projective line in the present article (see [1] to [9]).

There exist only two manifolds of dimension 1, in the topological sense: the open interval and the circle. We consider them separately.

Case A. Z *is topologically an interval*. Here $\tilde{Z}$ is projectively equivalent with an interval in $\tilde{P}$. The classification into projectively different cases is as follows:

A1. $Z = \tilde{P}$.

A2. Z is one of the two parts into which a point in $\tilde{P}$ divides its complement in $\tilde{P}$.

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A3. Z is an interval in $\tilde{P}$ which under the covering map: projection $\tilde{P} \to P$ covers some open interval in P n times, and its complement in P n-1 times, this complement being one point.

A4. Idem, but the complement being more than one point.

In the cases A3 and A4, the integer n is the only numerical projective invariant of Z.

Case B. Z is *topologically a circle*. The universal covering space $\tilde{Z}$ is an interval in $\tilde{P}$. The fundamental group (Poincaré group) operates in $\tilde{Z}$ as a group of projective transformations generated by one element, say h. This element h is represented under the covering map $\tilde{P} \to P$ by a projective transformation in P. The projective classification of projective orientation-preserving transformations of the real projective line is known. Each class is a property which a projective structure in a topological circle may or may not have. Canonical expressions in terms of nonhomogeneous coordinates are as follows:

B1. $x^* = r\,x, \quad r > 1;$

B2. $x^* = x + 1;$

B3. $x^* = \dfrac{x \cos \alpha + \sin \alpha}{-x \sin \alpha + \cos \alpha}$.

We now have to lift these projective transformations into $\tilde{P}$; in each case, this can be done in many ways. In the cases B1 and B2 there exists an invariant point $x = \infty$ in P. In the lifted transformation there will either be a corresponding invariant point (we will indicate this case by $n = 0$); or a corresponding point in $\tilde{P}$ and its image under the generator of the fundamental group will bound an interval, of the kind described in A3, with the invariant $n > 0$. The nonnegative integer n is a projective invariant of the space Z. If $n > 0$, then $\tilde{Z} = \tilde{P}$. If $n = 0$, then $\tilde{Z}$ can be identified as follows: in case B1, with the part $x > 0$ of the affine line; in case B2, with the affine line. In these cases, Z admits a compatible locally affine and a locally euclidean structure, respectively.

In case B3, it is advantageous to represent the projective line as the pencil of lines on a point in a euclidean plane, and the transformation B3 as induced by a rotation over an angle α of the plane about the center of the pencil. In the universal covering space $\tilde{P}$ we can then introduce a coordinate ϕ $(-\infty < \phi < \infty)$ which corresponds to angular measure in the euclidean plane. The coordinate system with coordinate ϕ covers $\tilde{P}$ completely, but it is not a projectively preferred coordinate system. In terms of these coordinates the generator of the fundamental group is expressed by

$$\phi \to \phi^* = \phi + \alpha.$$

A fundamental domain in $\tilde{P}$ is an interval, and it has that invariant n, defined in A3 and A4, which is determined by the relation

$$(n-1)\pi < \alpha \leq n\alpha.$$

We summarize our results:

THEOREM. *A complete classification of locally projective spaces of dimension one is given by* A1-A4, B1-B3 *and the invariant* n. *A space has a compatible locally* *affine structure in the cases* A3, A4, B1, B2, *provided* n = 0, *and in no other case. It has a compatible locally euclidean structure in the cases* A3, A4, B2, *provided* n = 0, *and in no other case.*

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