INVOLUTION AND EQUIVALENCE

by

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The attitude taken in what follows may be briefly characterized by saying that we want to see how far we can go in mathematics using only a minimal logical apparatus. We want to avoid using the concepts of set, element, existence, uniqueness, identity, and, if possible, negation. We do use the concept of relation and implication. In this sense one may discuss the minimal content of a mathematical theory. The procedure is to take a conventional mathematical theory, to express it in terms of the properties of a relation and then to consider only those properties that can be formulated in terms of implication.

This attitude is the same as that taken in a paper on ternary relations in the first volume of this Journal (pp. 97 to 111). The simplest example of a minimal theory in the above sense considered there is the customary definition of a equivalence relation. It consists of three propositions (p. 109):

1. (aa)'
2. (ab)' implies (ba)'
3. (ac)', (bc)' imply (ab)'

In this paper (ab)' is to be read: a and b are equivalent. We want to consider the minimal content of the concept of <u>involution</u> or involutory transformation, or of periodic transformation of period two. The conventional theory from which we start deals with a set S of elements x, y, z etc. To every element x is assigned uniquely an element y=f(x) and if y is assigned to x then x is assigned to y.

The first step consists in introducing instead of the operation y=f(x) a binary relation which we shall denote by (xy)*; at this stage we will say that (xy)* means that y is assigned to x: and since, as we said above, in this case x is assigned to y we can say that (xy)* implies (yx)*; in other words, the relation is symmetric. If it were only that we wanted to translate the original theory in terms of relations we would have to say that given x there exists a y such that (xy)*; and furthermore, that there is only one such y. The first part of this statement has no counterpart in our theory; the discussion of the second part we begin by reformulating it (still within the conventional theory) by saying that if there are two elements y and z each of which is in involution with x - they must be identical; in other words, that (xy)* and (xz)* imply that y is the same as z, or identical with z. But we do not want to use the concept of identity in the ultimate theory, so we replace the statement at this stage by a milder statement; namely, we'll say that if (xy)* and (xz)* then y and z are identical <u>relative to the re-</u>

lation ( )*; or explicitly, that whenever y satisfies with some u the relation (yu)* then z must satisfy the same relation (zu)*, that is that if (xy) * and (xz)* and also (yu)* then (zu)*.

Now we make the last step and write the last statement as a pure implication that is considering x, y, z, u as pure symbols rather than as symbols standing for the elements of a set. We thus write the proposition which in our theory is the counterpart of the uniqueness statement as

(xy)*, (xz)*, (yu)* imply (zu)*.

This proposition together with the statement of symmetry constitutes thus the minimum theory corresponding to the concept of involution. Before we go on we note that using symmetry we may write this proposition in a more suggestive form. The final formulation of our axioms is then

A. (xy)* implies (yx)*
B. (zx)*, (xy)*, (yu)* imply (zu)*.

Compare now the two sets of axioms: the one characterizing equivalence and the one characterizing involution. They have the symmetry property in common. Furthermore, proposition B, is similar to proposition 3. expressing transitivity; it is of the same type but it has, so to say, one more link. Proposition 3. does not follow from B. alone or for that matter from B. and symmetry (a counterexample is furnished by the involution y= -x, where x and y are integers). But if we adjoin to A. and B. the counterpart of 1., namely

C. (xx)*

then writing in B. x for z and dropping in view of C. the first assertion we get

(xy)*, (yu)* imply (xu)*

which is a counterpart of 3.

In addition to giving a new example of our attitude introduced in the paper mentioned above, and pointing out the unexpected similarity between equivalence relation and the concept of involution the above leads to the suggestion that the system of axioms consisting of A., B., and C. is preferable to 1., 2., 3. for characterizing equivalence relations because B. taken separately is weaker than the corresponding proposition 3.

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