A REMARK ON $\mathcal{L}^*$-SPACES

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An $\mathcal{L}^*$-space is one in which the notion of a convergent sequence is defined, such that (1) every subsequence of a convergent sequence converges to the same limit, (2) if $p_n \equiv p$, then $\mathrm{Lim}_{n\to\infty} p_n = p$, and (3) if it is false that $\mathrm{Lim}_{n\to\infty} p_n = p$, then the sequence $p_1, l_2, \ldots$ contains a subsequence no subsequence of which converges to p. [1, pp. 76-77.] The definition of convergent sequences induces a topology in such a space in a natural manner; and if the space is then a Hausdorff space, there are two possible ways of defining the topology of a Cartesian product $X \times Y$: (I) We can first define the topology in X and Y, and then define the product topology in the usual manner, or (II) we can say that $\mathrm{Lim}_{n\to\infty}(x_i, y_i) = (x, y)$ if and only if $\mathrm{Lim}_{n\to\infty} x_i = x$ and $\mathrm{Lim}_{n\to\infty} y_i = y$, and then define the topology in $X \times Y$ by means of the convergent sequences in $X \times Y$. The latter definition is the one adopted in [2]. The following simple example shows that if the first axiom of countability is not imposed, (I) and (II) may give different results, even if it is true that (II) makes $X \times Y$ a Hausdorff space. We shall call a Hausdorff space a convergence space if its topology can be induced by a definition of convergent sequences satisfying (1), and (2), and (3). Our example, the n, is of two convergence spaces whose product (in sense (II)) is a Hausdorff space and whose product (in sense (I)) is not a convergence space.

(1) Let $A = (a_{i,j})$ be an infinite matrix (i=1, 2, ...; j=1, 2, ...); let P be an ideal point; and let X be A+P. Let the neighborhoods in S_1 be the elements of

A and the sets P+M, where M is a set which contains all but a finite number of the elements of each row of A.

(2) Let $B = (b_{i,j})$ be an infinite matrix $(i = 1, 2, \ldots; j = 1, 2, \ldots)$; let Q be an ideal point; and let Y be B+Q. Let the neighborhoods in Y be the elements of B and the sets Q+N, where N is a set which contains all but a finite number of the rows of B.

It is clear that X and Y are convergence spaces.

Let D be the set of all elements of the product $X \times Y$ which have the form $(a_{i,j}, b_{i,j})$. It is easy to show that (P, Q) is a limit-point of D. But if $Z_1, Z_2, \ldots$ is a sequence of points of D having (P, Q) as a limit-point, then the sequence $x_1, x_2, \ldots$ obtained by projecting the terms of the Z-sequence onto **X** must have P as a limit-point, and hence must contain infinitely many elements of some row of A. This gives a divergent subsequence of the Z-sequence. Therefore no sequence of points of D converges to (P, Q); and $X \times Y$ is not a convergence space.

If the topology of X Y is defined as in (II), then the topology is discrete, so that the Hausdorff separation axiom is trivially satisfied.

Bibliography

[1] C. Kuratowski, Topologie I, first edition.

[2] C. Kuratowski, Sur la notion de limite topologique d'ensembles, Ann, Soc. Polon. Math. Vol. 21 (1948) p. 219.