ON THE SUMMABILITY OF ORDINARY DIRICHLET SERIES BY TAYLOR METHODS

by

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For any real constant α in the interval $0 \le \alpha \le 1$, the symbol T_α shall denote the regular sequence-to-sequence transformation represented by the upper-triangular Toeplitz matrix $(t_{n,k})$, where

$$t_{nk} = (1 - \alpha)^{n+1} C_{k,n} \alpha^{k-n}$$

for $n = 0, 1, \ldots$ and $k \ge n$, and $t_{nk} = 0$ for $k < n$. The transformations T_α were introduced as "circle methods" by G.H. Hardy and J.E. Littlewood [2] in connection with a certain Tauberian theorem on the Borel transformation. R. Wais [6] and W. Meyer-König [3] made extensive investigations concerning the application of these transformations to Taylor series, and they introduced the name Taylor-Verfahren. The transformations T_α were again introduced, independently and without the restriction of α to real values, by P. Vermes [4], [5], and by V. F. Cowling [1].

In the present paper we prove two theorems concerning Taylor transformations of ordinary Dirichlet series. It is convenient to replace the transformations T_α by the corresponding series-to-series transformations V_α :

$$V_\alpha \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n ,$$

where

$$b_n = \sum_{k=n}^{\infty} v_{nk} a_k \text{ and } v_{nk} = (1 - \alpha)^n C_{k,n} \alpha^{k-n}$$

for $k \ge n$, $v_{n,k} = 0$ for $k < n$. If the V_α transform of

$\sum a_n$ converges absolutely, the series $\sum a_n$ is said to be absolutely summable by V_α.

Theorem 1. Let $\{a_n\}$ be a sequence of complex numbers with the property $\limsup |a_n|^{1/n} = 1/R < \infty$, and let α be a real number $(0 \leq \alpha < 1/3, \alpha < R)$. If the series $\sum a_n (n+1)^{-s}$ is summable by V_α at the point $s = s_0$, it is absolutely summable by V_α throughout the half-plane $\Re s > 1 + \Re s_0$.

Theorem 2. If under the conditions of Theorem 1 the series $\sum a_n (n+1)^{-s}$ is absolutely summable by V_α at the point s_0, it is absolutely summable by V_α throughout the half-plane $\Re s > \Re s_0$.

Up to a certain stage, the proofs of the two theorems are identical. Since V_0 is the identity transformation for series, it may be assumed, in the proof, that α is positive.

Lemma. If $0 < \alpha < 1/2$, $\limsup |a_n|^{1/n} < 1/\alpha$, and the series $\sum b_n$ converges, where

$$b_n = (1 - \alpha)^n \sum_{k=n}^{\infty} C_{k,n}\, \alpha^{k-n} a_k$$

for $n = 0, 1, \ldots$, then

$$a_k = (1 - \alpha)^{-k} \sum_{n=k}^{\infty} C_{n,k}[-\alpha/(1-\alpha)]^{n-k} b_n$$

for $k = 0, 1, \ldots$.

This lemma is suggested by the fact that the matrix T_α has the formal inverse $T_{-\alpha/(1-\alpha)}$. It is a consequence of the following considerations. If $f(z) = \sum a_n z^n$, then

$$f^n(\alpha) = \sum_{m=n}^{\infty} a_m\, \alpha^{m-n}\, m!/(m-n)!,$$

and therefore

$$f(z) = \sum_{n=0}^{\infty} f^n(\alpha)(z-\alpha)^n/n!$$

$$= \sum_{n=0}^{\infty} (z-\alpha)^n \sum_{m=n}^{\infty} C_{m,n}\alpha^{m-n} a_m$$

inside of the circle of holomorphism of f(z) about the point $z = \alpha$. Convergence of $\sum b_n$ therefore implies that the Taylor series of f(z) about $z = \alpha$ converges at the point z = 1. This in turn implies that the Taylor series converges in a neighborhood of the origin, and therefore that

$$a_k = f^k(0)/k!$$

$$= \sum_{n=k}^{\infty}(0-\alpha)^{n-k} C_{n,k} \sum_{m=n}^{\infty} C_{m,n}\alpha^{m-n} a_m$$

$$= \sum_{n=k}^{\infty}(-\alpha)^{n-k} C_{n,k}(1-\alpha)^{-n} b_n \quad ,$$

and the lemma is proved.

To proceed with the proof of the theorems, let $\sum b_n$ be the V_α transform of the series $\sum a_m(m+1)^{-s_0}$. The application of the lemma gives the result

$$a_k(k+1)^{-s_0} = \sum_{j=k}^{\infty} C_{j,k}(-\alpha)^{j-k}(1-\alpha)^{-j} b_j ,$$

and the series whose absolute convergence is to be established can therefore be written in the form

$$(1) \quad \sum_{n=0}^{\infty}(1-\alpha)^n \sum_{k=n}^{\infty} C_{k,n}\alpha^{k-n} a_k(k+1)^{-s} = \sum_{n=0}^{\infty} p^{-n} B_n ,$$

where $p = \alpha/(1-\alpha)$ and

$$(2) \quad B_n = \sum_{k=n}^{\infty} C_{k,n}(k+1)^{-(s-s_0)} \sum_{j=k}^{\infty}(-1)^{j-k} C_{j,k} p^j b_j .$$

Formally, reversing the order of summation in the right member of (2) gives the result

(3) $$B_n = \sum_{j=n}^{\infty} p^j b_j \sum_{k=n}^{j} C_{k,n}(k+1)^{-(s-s_o)}(-1)^{j-k} C_{j,k}.$$

Because $C_{k,n}C_{j,k} = C_{j,n}C_{j-n,k-n}$ and (for $\mathfrak{R}s > \mathfrak{R}s_o$)

$$(k+1)^{-(s-s_o)} = \frac{1}{\Gamma(s-s_o)} \int_0^{\infty} t^{s-s_o-1} e^{-(k+1)t} dt,$$

equation (3) can (with $s - s_o - 1 = z = x + iy$) be written in the form

(4) $$B_n = \frac{1}{\Gamma(s-s_o)} \sum_{j=n}^{\infty} p^j b_j (-1)^{j-n} C_{j,n} \cdot \int_0^{\infty} t^z e^{-(n+1)t}(1-e^{-t})^{j-n} dt.$$

Moreover, the sum of the absolute values of the terms in the double series of (3) is not greater than

$$\frac{1}{\Gamma(s-s_o)} \sum_{j=n}^{\infty} p^j |b_j| C_{j,n} \int_0^{\infty} t^x e^{-(n+1)t}(1+e^{-t})^{j-n} dt,$$

and this is finite because $b_j \to 0$, the integral involved is not greater than $2^{j-n} \int_0^{\infty} t^x e^{-t} dt$, and $p < 1/2$ when $\alpha < 1/3$. It follows that the right member of (2) converges absolutely and is equal to the right member of (4). The theorems will therefore be proved when it is shown that the series

(5) $$\sum p^{-n} B_n \Gamma(s - s_o)$$

converges absolutely, where the symbol B_n represents the right member of (4).

Under the summation sign in (4), let the factor $b_j(-1)^{j-n}$ be replaced by $|b_j|$, and let the integrand be replaced by its absolute value. Let the resulting expression be substituted for B_n in the series (5); and let the order of summation in the result of this substitution be reversed. The resulting series has the form

$$\sum_{j=0}^{\infty} |b_j| \sum_{n=0}^{j} p^{j-n} C_{j,n} \int_0^{\infty} t^x e^{-(n+1)t} (1-e^{-t})^{j-n}\, dt.$$

Since the order of the second summation sign and the integral sign may be reversed with impunity, this can be simplified to the form

$$(6) \quad \sum_{j=0}^{\infty} |b_j| \int_0^{\infty} t^x e^{-t} [e^{-t} + p(1 - e^{-t})]^j\, dt,$$

and the theorems will be proved when it is shown that the series (6) converges absolutely.

Theorem 2 will be dispatched first. Since the quantity in brackets is a decreasing function of t, for positive values of t, its value is less than 1, for positive values of t. It follows that the integral has an upper bound independent of j, and the absolute convergence of the series $\sum b_j$ guarantees the absolute convergence of the series (6).

In the proof of Theorem 1, we use slightly less than the hypothesis that the series $\sum b_j$ converges; it is sufficient to assume that the terms of the series are bounded. For the integral in (6) can be written

$$\left(\int_0^a + \int_a^{\infty}\right) t^x e^{-t} [e^{-t} + p(1-e^{-t})]^j\, dt = I_j + I_j'.$$

Since

$$I_j' < [e^{-a} + p(1-e^{-a})]^j \int_0^{\infty} t^x e^{-t}\, dt,$$

and series $\sum b_j I_j'$ converges absolutely. On the other hand

$$I_j = \int_b^1 |\log u|^x [u(1 - p) + p]^j\, du,$$

where $b = e^{-a}$. The substitution $u = 1 - v$, together with the inequality $|\log (1 - v)| \leq 2v$ (valid throughout the range of integration, if b is sufficiently near to 1) gives the estimate

$$I_j < \int_0^{1-b} (2v)^x [1 - (1 - p)v]^j \, dv$$

$$= 2^x(1-p)^{-1-x} \int_0^{(1-b)(1-p)} w^x(1-w)^j \, dw$$

$$< c \int_0^1 w^x(1-w)^j \, dw = 0\,(j^{-1-x}) ,$$

and therefore the series $\sum b_j I_j$ also converges absolutely. It follows that the series (6) converges absolutely, and the proof is complete.

References

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