NORMED FIELDS OVER THE REAL AND COMPLEX FIELDS [1)]

by Leonard Tornheim

A normed field F over a field K, where K is contained in the complex field, is a field containing K and for which there is a real valued function $\|y\|$ called a norm satisfying the following conditions:

(1) $\|y\| > 0$ if $y \neq 0$,

(2) $\|y + z\| \leqq \|y\| + \|z\|$,

(3) $\|yz\| \leqq \|y\| \cdot \|z\|$,

(4) $\|cy\| = |c| \, \|y\|$,

where y, z are in F and c is in K.

We shall prove the following results:

THEOREM. Every normed field over the real field R is either the real field R or the complex field C.

COROLLARY. The complex field is the only normed field over the complex field.

These results are not new. Closely related theorems have been stated or proved by Mażur, Gelfand, Arens, Kaplansky, and Ramaswami (see bibliography). But all of their proofs use complex variable theory. Here, however, no use is made of the theory of functions of a complex variable nor of the completion of F. Ostrowski has proved the weaker theorem which has equality in (3).

LEMMA 1. If $\|y\| \leqq 1/2$, then $\|1/(1-y)\| \leqq 2\|1\|$ wherever $1/(1-y)$ exists.

For, $1/(1 - y) = 1 + y/(1 - y)$. Hence

1) Presented to the American Mathematical Society on September 7, 1951.

$$\|1/(1-y)\| \leq \|1\| + \|y\| \cdot \|1/(1-y)\|.$$

Thus for $\|y\| \leq 1/2$, $\|1/(1-y)\| < 2\,\|1\|$.

LEMMA 2. The function $1/y$ is continuous wherever it is defined.

For $\|1/y - 1/z\| <$

$$\|z-y\| \cdot \|1/z\|^2 \cdot \left\| \frac{1}{1-(z-y)/z} \right\|.$$

Hence as $\|z-y\| \to 0$, the right hand side goes to 0, where we have used (2), Lemma 1, and the fact that $\|(z-y)/z\| \leq \|z-y\| \cdot \|1/z\|$.

LEMMA 3. The following functions are continuous: $\|x\|$, $x+y$, xy.

These are well know facts. The first follows from $\big|\,\|y\| - \|z\|\,\big| \leq \|y-z\|$, a consequence of (2). The proof of the second also uses (2). Finally $\|yz - y_1z_1\| \leq \|y\|\,\|z-z_1\| + (\|z\| + \|z_1 - z\|)\,\|y - y_1\|$ which goes to 0 as $\|y - y_1\|$ and $\|z - z_1\|$ go to 0.

We shall give a direct proof of the corollary first in order to illuminate the proof of the theorem. Hence we now assume K = C, the complex field.

We shall show that for any y in F there is a value of c in C for which y - c does not have a reciprocal. Then, since F is a field, y - c = 0 and y = c is in C.

LEMMA 4. For any y in F having a reciprocal there is a value k such that $\|1/(y-c)\| < \|1/y\|$ for every $|c| > k$ for which $1/(y-c)$ exists.

As $|c| \to \infty$, $\|1/(y-c)\| = |-1/c| \cdot \|1/(1-y/c)\| \to 0$ by Lemma 1 since $\|y/c\| = |1/c| \cdot \|y\| \to 0$.

We now assume that y - c has a reciprocal for every c in C. This will lead to a contradiction.

Now $\|c\|$ is a continous function of c (using absolute value to measure distance) since $\|c\| = |c| \cdot \|1\|$. Hence $\|1/(y - c)\|$ is a continuous function, being a continuous function of a continuous function. Therefor $\|1/(y - c)\|$ attains a maximum value M on the bounded and closed set $|c| \leqq k$, where k is given by Lemma 4. This value M is not attained for $|c| > k$ since then $\|1/(y - c)\| < \|1/(y - 0)\| \leqq M$ by Lemma 4. Let C_0 be the set of all c for which $\|1/(y - c)\| = M$. Then C_0 is closed, bounded, and non-vacuous.

Let c_0 be a boundary point of C_0; such a c_0 exists. There also exists a value d such that d is not in C_0 but $r = |d - c_0| < 1/M$. Let S be the circle of radius r and center c_0, i.e., the set of values c for which $|c - c_0| = r$.

On S we take the n equally spaced points given by $c_0 + ru^v (v = 1, \ldots, n)$ where u is the n-th root of unity $e^{2\pi i/n}$. Let

$$S(n) = \frac{1}{n} \sum_{v=1}^{n} \frac{1}{y - (c_0 + ru^v)}.$$

If $f(x) = x^n - r^n = \prod_{v=1}^{n} (x - ru^v)$, the logarithmic derivative is $nx^{n-1}/(x^n - r^n)$ and equals $\sum_{v=1}^{n} 1/(x - ru^v)$. Setting $x = y - c_0 = z$ we see that

$$S(n) = z^{n-1}/(z^n - r^n)$$
$$= 1/[z - r(r/z)^{n-1}].$$

This expression has meaning since $1/(z^n - r^n) = \prod 1/(z - ru^v)$. Hence by Lemma 2,

$$\lim_{n\to\infty} \|S(n)\| = \|1/z\| = M \tag{5}$$

since $\|r(r/z)^{n-1}\| \leqq r(r\|1/z\|)^{n-1} \to 0$

because $r < 1/M = \|1/z\|^{-1}$.

Now d lies on the circle S and not in the closed set $S \cap C_0$. Therefore d lies on a non-vacuous open arc of $S - C_0$ and hence on a closed arc S' in $S - C_0$ of length $2\pi r/m$, i.e., 1/m-th of the circumference, for m a sufficiently large integer. Hence if mq points are taken on S equally spaced, q of them will lie on S'. Because S' is closed and bounded $\|1/(y - c)\|$ assumes a maximum value M' for s in S', and $M' < M$ since $S' \cap C_0 = \emptyset$.

Then since q values of $c_0 + ru^v$ lie in S',

$$\|S(mq)\| \leqq \frac{1}{mq}[qM' + (mq - q)M]$$

by (2) and (4). This gives

$$\|S(mq)\| \leqq M - (M - M')/m$$

so that $\|S(mq)\|$ is bounded away from M for all values of q. This result is a contradicition to (5). Hence we cannot assume that y - c always has a reciprocal. The proof of the corollary is completed.

The theorem can be proved from the corollary by extending the norm to F(i) and then introducing a new norm in which (4) is true for c in R(i) [Kaplansky, 4, p. 400]. Instead we now prove the theorem following the pattern of the proof of the corollary. Let K be the real field R. We shall show that for any y in F there are values of r_1, r_2 in R for which $y^2 + r_1 y + r_2$ does not have a reciprocal. Since F is a field, this expression is 0 and y is in the complex field C.

We assume that $y^2 + r_1 y + r_2$ has a reciprocal for all pairs r_1, r_2 in R, and we shall arrive at a contradiction. Let G be the field R(y, i). In G the norm $\|z\|$ is defined only for those elements z which lie in F, and thus for all elements in R(y). In particular if $c = a + bi$ is in C and $\overline{c}$ is the complex conjugate of c, $\| 1/(y - c) + 1/(y - \overline{c}) \| = \| 2(y-a)/[(y-a)^2 + b^2] \|$ and is defined. This expression is a continuous function of a and b by Lemmas 2 and 3 and the fact that a continuous function of a continuous function is continuous.

LEMMA 5. The expression $\| 1/(y - c) + 1(y - \overline{c}) \|$ is a continuous function of c.

This follows from the discussion immediately above and the fact that a and b are continuous functions of c.

LEMMA 6. There exists a value of k such that $\| 1/(y - c) + 1/(y - \overline{c}) \| < \| 2/y \|$ whenever $|c| > k$.

For

$$1/(y - c) + 1/(y - \overline{c}) = \frac{2y/c\overline{c} - [1/c + 1/\overline{c}]}{1 - y[1/c + 1/\overline{c}] + y^2/c\overline{c}}$$

and as $|c| \to \infty$,

$$\| 2y/c\overline{c} \| = |1/c\overline{c}| \cdot \| 2y \| \to 0$$

$$\| 1/c + 1/\overline{c} \| = |1/c + 1/\overline{c}| \cdot \| 1 \| \to 0$$

$$\| y[1/c + 1/\overline{c}] \| = |1/c + 1/\overline{c}| \cdot \| y \| \to 0$$

$$\| y^2/c\overline{c} \| = |1/c\overline{c}| \cdot \| y^2 \| \to 0,$$

so that by Lemmas 2 and 3,

$$\| 1/(y - c) + 1/(y - \overline{c}) \| \to 0.$$

This implies the lemma.

From Lemmas 5 and 6 it follows that $\| 1/(y - c) + 1/(y - \overline{c}) \|$ attains its maximum value M for c in C, and the set C_0 on which this occurs is closed and bounded. Incidentally C_0 is symmetric with respect to the real axis.

There exists a boundary point c_0 of C_0, and also a point d not in C_0 with $r = |d - c_0| < \min(N, L^{1/2})$ where $L = \| 1/(y - c_0)(y - \overline{c}_0) \|$ and $N = \| 1/(y - c_0) \| + \| 1/(y - \overline{c}_0) \|$. Let S be the circle of radius r and center c_0. On S take the n equally spaced points given by $c_0 + ru^v$ $(v = 1, \ldots, n)$, where $u = e^{2\pi i/n}$. Let

$$S(n) = \frac{1}{n} \sum_{v=1}^{n} \left[\frac{1}{y - (c_0 + ru^v)} + \frac{1}{y - \overline{(c_0 + ru^v)}} \right].$$

Let $z = y - c_0$ and $\overline{z} = y - \overline{c}_0$. Then

$$S(n) = 1/(z - r(r/z)^{n-1}) + 1/(\overline{z} - r(r/\overline{z})^{n-1})$$

$$= \frac{z + \overline{z} - r^n T_{n-1}}{z\overline{z}(1 - r^n T_n) + r^{2n}(z\overline{z})^{-n+1}}$$

where $T_n = z^{-n} + \overline{z}^{-n}$. Now $M = \| T_1 \|$. To show that $\lim_{n\to\infty} \| S_n \| = M$, it is sufficient to show that the norm of each term involving n has limit 0. First

$$\| T_n \| \leqq \| z^{-1} \|^n + \| \overline{z}^{-1} \|^n$$

$$\leqq (\| z^{-1} \| + \| \overline{z}^{-1} \|)^n.$$

Hence $\| r^n T_n \| \leqq (rN)^{n-1}$ and has limit 0 since $r < N$. Next, $\| r^{2n}(z\overline{z})^{-n} \| \leqq (r^2 L)^n$ and also has limit 0 since $r < L^{1/2}$. Hence $\lim_{n\to\infty} \| S_n \| = M$.

The rest of the proof, that on the other hand $\| S(mq) \| \leqq M - (M - M')/m$, is the same as that of the last two paragraphs in the proof of the corollary except that M' is the maximum of $\| 1/(y - c) + 1/(y - \bar{c}) \|$ for c in S'. The contradiction arrived at here permits one to conclude that $y^2 + r_1 y + r_2$ does not always have a reciprocal and must be 0 for suitable real r_1, r_2 since F is a field. Thus y is in C.

COROLLARY 2. <u>The only normed division algebra over the complex field is the complex field.</u>

By a normed division algebra Q over C is meant a division algebra over C with a norm satisfying (1), (2), (3), (4). Then if y is in D, R(y) is a normed field over C. Hence R(y) = C and y is in C, by Corollary 1.

COROLLARY 3. <u>Every normed division algebra over the real field is either the real field, the complex field, or the real quaternion algebra.</u>

The proof given by Arens [1, p. 626] applies here since we have indirectly shown that every element of the division algebra satisfies a quadratic equation with real coefficients.

Bibliography

1. R. Arens, Linear topological division algebras, Bull. Amer. Math. Soc. 53, 623-630 (1947).

2. I. Gelfand, Normierte Ringe, Mat. Sbornik N. S. 9, 3-24 (1941).

3. I. Kaplansky, Topological rings, Bull. Amer. Math. Soc. 54, 809-826 (1948).

4. I. Kaplansky, Normed algebras, Duke Math. J. 16, 399-418 (1949).

5. S. Mazur, Sur les anneaux lineaires, C. R. Acad. Sci. Paris 207, 1025-1027 (1938).

6. A. Ostrowski, Über einige Lösungen der Funktionalgleichung $\phi(x)\cdot\phi(y) = \phi(xy)$, Acta Math. 41, 271-284 (1917).

7. V. Ramaswami, Normed algebras, isomorphism and the associative postulate, J. Indian Math. Soc. (N. S.) 14, 47-64 (1950).