AMBIGUOUS POINTS OF A FUNCTION HARMONIC INSIDE A SPHERE

F. Bagemihl

Let x, y, z denote the Cartesian coordinates of a point in three-dimensional Euclidean space, and set

$$S = \{(x, y, z): x^2 + y^2 + z^2 < 1\}, \qquad T = \{(x, y, z): x^2 + y^2 + z^2 = 1\}.$$

THEOREM. *There exists a harmonic function* $h(P)$ $(P \in S)$ *such that, for every* $Q \in T$ *and every real number* r, *including the values* $+\infty$ *and* $-\infty$, *there is a Jordan arc* J_r^Q *lying wholly in* S *except for its end point* Q, *with the property that*

$$\lim_{P \to Q, P \in J_r^Q} h(P) = r.$$

Proof. Piranian has shown [2, Remark 2] that there exists a continuous function $f(P)$ $(P \in S)$ such that the assertion we have made concerning the boundary behavior of $h(P)$ holds for $f(P)$; he has constructed a tree G in S such that, for every $Q \in T$ and every r, J_r^Q is, except for its end point Q, a subarc of G. To define $h(P)$, we shall make use of G and $f(P)$.

Let

$$0 < r_0 < r_1 < \cdots < r_n < \cdots < 1, \qquad \lim_{n \to \infty} r_n = 1,$$

$$S_n = \{(x, y, z): x^2 + y^2 + z^2 < r_n^2\}, \qquad T_n = \{(x, y, z): x^2 + y^2 + z^2 = r_n^2\} \qquad (n = 0, 1, 2, \cdots),$$

$$K_n = (S_n \cup T_n \cup G) \cap (S_{n+1} \cup T_{n+1}) \qquad (n = 0, 1, 2, \cdots).$$

For every nonnegative integer n, K_n is a compact set with the property that any continuous function on K_n that is harmonic at every interior point of K_n can be uniformly approximated on K_n as closely as desired by a harmonic polynomial (see [1]; I am indebted to Professor J. L. Walsh for this reference).

We define, by induction on n, a harmonic polynomial $h_n(P)$, as follows. Let

$$g_0(P) = 0 \qquad (P \in S_0 \cup T_0),$$

$$g_0(P) = f(P) \qquad (P \in G \cap T_1),$$

and let $g_0(P)$ be linear on each segment of G which extends from T_0 to T_1. Then $g_0(P)$ is continuous on K_0 and harmonic at every interior point of K_0, and hence there exists a harmonic polynomial $h_0(P)$ for which

$$|h_0(P) - g_0(P)| < 1 \qquad (P \in K_0).$$

Suppose that $n > 0$, and that we have defined the harmonic polynomial $h_{n-1}(P)$. Let

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$$g_n(P) = h_{n-1}(P) \qquad (P \in S_n \cup T_n),$$

$$g_n(P) = f(P) \qquad (P \in G \cap T_{n+1}),$$

and let $g_n(P)$ be linear on each segment of G which extends from T_n to T_{n+1}. Then $g_n(P)$ is continuous on K_n and harmonic at every interior point of K_n, and hence there exists a harmonic polynomial $h_n(P)$ for which

$$|h_n(P) - g_n(P)| < 2^{-n} \qquad (P \in K_n).$$

This completes the induction.

If $j \geq k \geq n \geq 1$, then

$$|h_j(P) - h_k(P)| < \sum_{m=k+1}^{\infty} 2^{-m} \qquad (P \in S_n \cup T_n),$$

so that the sequence $\{h_n(P)\}$ converges uniformly on every compact subset of S to a harmonic function $h(P)$ $(P \in S)$.

If $P \in G \cap (S_{n+1} - S_n)$, where $n \geq 1$, then

$$|h(P) - f(P)| < \sum_{m=n-1}^{\infty} 2^{-m}.$$

Consequently,

$$\lim_{P \to T, P \in G} [h(P) - f(P)] = 0,$$

and the proof of our theorem is complete.

REFERENCES

1. J. Deny, *Sur l'approximation des fonctions harmoniques*, Bull. Soc. Math. France 73 (1945), 71-73.

2. G. Piranian, *Ambiguous points of a function continuous inside a sphere*, Michigan Math. J. 4 (1957), 151-152.

University of Notre Dame